35. If $a_n = \dfrac{x^n}{n^2}$, then by the Ratio Test, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{x^n}\right| = |x|\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^2 = |x| < 1$ for convergence, so $R = 1$. When $x = \pm 1$, $\sum\limits_{n=1}^{\infty}\left|\dfrac{x^n}{n^2}\right| = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ which is a convergent p-series ($p = 2 > 1$), so the interval of convergence for f is $[-1,1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need only check the endpoints. $f(x) = \sum\limits_{n=1}^{\infty}\dfrac{x^n}{n^2} \Rightarrow f'(x) = \sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n^2} = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n+1}$, and this series diverges for $x = 1$ (harmonic series) and converges for $x = -1$ (Alternating Series Test), so the interval of convergence is $[-1,1)$. $f''(x) = \sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim\limits_{n\to\infty}\dfrac{n}{n+1} = 1 \neq 0$, so its interval of convergence is $(-1,1)$.

36. (a) $\sum\limits_{n=1}^{\infty} nx^{n-1} = \sum\limits_{n=0}^{\infty}\dfrac{d}{dx}x^n = \dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}x_n\right] = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = -\dfrac{1}{(1-x)^2}(-1) = \dfrac{1}{(1-x)^2}$, $|x| < 1$.

(b) (i) $\sum\limits_{n=1}^{\infty} nx^n = x\sum\limits_{n=1}^{\infty} nx^{n-1} = x\left[\dfrac{1}{(1-x)^2}\right]$ [from part (a)] $= \dfrac{x}{(1-x)^2}$ for $|x| < 1$.

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n} = \sum\limits_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \dfrac{1/2}{(1-1/2)^2} = 2$.

(c) (i) $\sum\limits_{n=2}^{\infty} n(n-1)x^n = x^2\sum\limits_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\dfrac{d}{dx}\left[\sum\limits_{n=1}^{\infty} nx^{n-1}\right] = x^2\dfrac{d}{dx}\dfrac{1}{(1-x)^2}$

$$= x^2\frac{2}{(1-x)^3} = \frac{2x^2}{(1-x)^3} \text{ for } |x| < 1.$$

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=2}^{\infty}\dfrac{n^2-n}{2^n} = \sum\limits_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n = \dfrac{2(1/2)^2}{(1-1/2)^3} = 4$.

(iii) From (b)(ii) and (c)(ii), we have $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{2^n} = \sum\limits_{n=1}^{\infty}\dfrac{n^2-n}{2^n} + \sum\limits_{n=1}^{\infty}\dfrac{n}{2^n} = 4 + 2 = 6$.

37. By Example 7, $\tan^{-1}x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$. In particular, for $x = \dfrac{1}{\sqrt{3}}$, we have $\dfrac{\pi}{6} = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty}(-1)^n\left(\dfrac{1}{3}\right)^n\dfrac{1}{\sqrt{3}}\dfrac{1}{2n+1}$, so

$$\pi = \frac{6}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n}.$$

38. (a) $\displaystyle\int_0^{1/2}\frac{dx}{x^2-x+1} = \int_0^{1/2}\frac{dx}{(x-1/2)^2+3/4}\quad\left[\begin{array}{l} x-1/2 = (\sqrt{3}/2)u,\ u = (2/\sqrt{3})(x-1/2) \\ dx = (\sqrt{3}/2)\,du \end{array}\right]$

$$= \int_{-1/\sqrt{3}}^{0}\frac{(\sqrt{3}/2)\,du}{(3/4)(u^2+1)} = \frac{2\sqrt{3}}{3}\left[\tan^{-1}u\right]_{-1/\sqrt{3}}^{0} = \frac{2}{\sqrt{3}}\left[0-\left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{3\sqrt{3}}$$

(b) $\dfrac{1}{x^3+1} = \dfrac{1}{(x+1)(x^2-x+1)} \Rightarrow$

$$\frac{1}{x^2-x+1} = (x+1)\left(\frac{1}{1+x^3}\right) = (x+1)\frac{1}{1-(-x^3)} = (x+1)\sum_{n=0}^{\infty}(-1)^n x^{3n}$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{3n+1} + \sum_{n=0}^{\infty}(-1)^n x^{3n} \text{ for } |x| < 1 \Rightarrow$$

$$\int\frac{dx}{x^2-x+1} = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+2}}{3n+2} + \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+1}}{3n+1} \text{ for } |x| < 1 \Rightarrow$$

$$\int_0^{1/2}\frac{dx}{x^2-x+1} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{4\cdot 8^n(3n+2)} + \frac{1}{2\cdot 8^n(3n+1)}\right] = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\left(\frac{2}{3n+1} + \frac{1}{3n+2}\right).$$

By part (a), this equals $\dfrac{\pi}{3\sqrt{3}}$, so $\pi = \dfrac{3\sqrt{3}}{4}\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{8^n}\left(\dfrac{2}{3n+1} + \dfrac{1}{3n+2}\right)$.

8.7 Taylor and Maclaurin Series

1. Using Theorem 5 with $\sum\limits_{n=0}^{\infty} b_n(x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. (a) Using Formula 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $1.6 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

3. Since $f^{(n)}(0) = (n+1)!$, Equation 7 gives the Maclaurin series

$\sum\limits_{n=0}^{\infty}\dfrac{f^{(n)}(0)}{n!}x^n = \sum\limits_{n=0}^{\infty}\dfrac{(n+1)!}{n!}x^n = \sum\limits_{n=0}^{\infty}(n+1)x^n$. Applying the Ratio Test with $a_n = (n+1)x^n$ gives us

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x|\lim\limits_{n\to\infty}\dfrac{n+2}{n+1} = |x|\cdot 1 = |x|$. For convergence, we must have $|x| < 1$, so the radius of convergence $R = 1$.

4. Since $f^{(n)}(4) = \dfrac{(-1)^n n!}{3^n(n+1)}$, Equation 6 gives the Taylor series

$\sum\limits_{n=0}^{\infty}\dfrac{f^{(n)}(4)}{n!}(x-4)^n = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n n!}{3^n(n+1)\,n!}(x-4)^n = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{3^n(n+1)}(x-4)^n$, which is the Taylor series for f centered at 4. Apply the Ratio Test to find the radius of convergence R.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)}\cdot\frac{3^n(n+1)}{(-1)^n(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-4)(n+1)}{3(n+2)}\right|$$

$$= \frac{1}{3}|x-4|\lim_{n\to\infty}\frac{n+1}{n+2} = \frac{1}{3}|x-4|$$

For convergence, $\frac{1}{3}|x-4| < 1 \Leftrightarrow |x-4| < 3$, so $R = 3$.

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
⋮	⋮	⋮

We use Equation 7 with $f(x) = \cos x$.

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

If $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = x^2 \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x. \text{ So } R = \infty \text{ (Ratio Test).}$$

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2 \sin 2x$	0
3	$-2^3 \cos 2x$	-2^3
4	$2^4 \sin 2x$	0
⋮	⋮	⋮

$f^{(n)}(0) = 0$ if n is even and $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$, so

$$\sin 2x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^2|x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x,$$

so $R = \infty$ (Ratio Test).

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	$5^2 e^{5x}$	25
3	$5^3 e^{5x}$	125
4	$5^4 e^{5x}$	625
⋮	⋮	⋮

$$e^{5x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{5^n}{n!}x^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n|x|^n}\right]$$

$$= \lim_{n\to\infty}\frac{5|x|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
⋮	⋮	⋮

$$xe^x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}.$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x|^{n+1}}{n!}\cdot\frac{(n-1)!}{|x|^n}\right] = \lim_{n\to\infty}\frac{|x|}{n} = 0 < 1$ for all x, so $R = \infty$.

9.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1 + x + x^2$	7
1	$1 + 2x$	5
2	2	2
3	0	0
4	0	0
⋮	⋮	⋮

$$f(x) = 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 + \sum_{n=3}^{\infty}\frac{0}{n!}(x-2)^n$$

$$= 7 + 5(x-2) + (x-2)^2$$

Since $a_n = 0$ for large n, $R = \infty$.

10.

n	$f^{(n)}(x)$	$f^{(n)}(-1)$
0	x^3	-1
1	$3x^2$	3
2	$6x$	-6
3	6	6
4	0	0
5	0	0
⋮	⋮	⋮

$$f(x) = -1 + 3(x+1) - \tfrac{6}{2!}(x+1)^2 + \tfrac{6}{3!}(x+1)^3$$

$$= -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$$

Since $a_n = 0$ for large n, $R = \infty$.

11. Clearly, $f^{(n)}(x) = e^x$, so $f^{(n)}(3) = e^3$ and $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n$. If $a_n = \dfrac{e^3}{n!}(x-3)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

12.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3\cdot 2x^{-4}$	$-\frac{3\cdot 2}{16}$
⋮	⋮	⋮

$f^{(n)}(2) = \dfrac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \ge 1$, so

$$\ln x = \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-2)^n}{n\cdot 2^n}.$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-2|}{2} < 1$ for convergence, so

$|x-2| < 2 \;\Rightarrow\; R = 2.$

13.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
⋮	⋮	⋮

$$\cos x = \sum_{k=0}^{\infty}\frac{f^{(k)}(\pi)}{k!}(x-\pi)^k$$

$$= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x-\pi|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi|^{2n}}\right] = \lim_{n\to\infty}\frac{|x-\pi|^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

14.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
⋮	⋮	⋮

$$\sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!}\left(x-\frac{\pi}{2}\right)^k = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{(x-\pi/2)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x-\pi/2|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi/2|^{2n}}\right] = \lim_{n\to\infty}\frac{|x-\pi/2|^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x, \text{ so } R=\infty.$$

15.

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{-1/2}$	$\frac{1}{3}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2}\cdot\frac{1}{3^3}$
2	$\frac{3}{4}x^{-5/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{3^5}$
3	$-\frac{15}{8}x^{-7/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{3^7}$
⋮	⋮	⋮

$$\frac{1}{\sqrt{x}} = \frac{1}{3} - \frac{1}{2\cdot 3^3}(x-9) + \frac{3}{2^2\cdot 3^5}\frac{(x-9)^2}{2!} - \frac{3\cdot 5}{2^3\cdot 3^7}\frac{(x-9)^3}{3!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)[2(n+1)-1]\,|x-9|^{n+1}}{2^{n+1}\cdot 3^{[2(n+1)+1]}\cdot(n+1)!}\cdot\frac{2^n\cdot 3^{2n+1}\cdot n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,|x-9|^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(2n+1)\,|x-9|}{2\cdot 3^2(n+1)}\right] = \frac{1}{9}|x-9| < 1$$

for convergence, so $|x-9| < 9$ and $R = 9$.

16.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	-24
4	$120x^{-6}$	120
⋮	⋮	⋮

$$x^{-2} = 1 - 2(x-1) + 6\cdot\frac{(x-1)^2}{2!} - 24\cdot\frac{(x-1)^3}{3!} + 120\cdot\frac{(x-1)^4}{4!} - \cdots = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - \cdots = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)\,|x-1|^{n+1}}{(n+1)\,|x-1|^n} = \lim_{n\to\infty}\left[\frac{n+2}{n+1}\cdot|x-1|\right] = |x-1| < 1 \text{ for convergence, so } R=1.$$

17. If $f(x) = \cos x$, then $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so by Formula 9 with $a=0$ and $M=1$, $|R_n(x)| \le \frac{1}{(n+1)!}|x|^{n+1}$. Thus, $|R_n(x)| \to 0$ as $n\to\infty$ by Equation 10. So $\lim_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 5 represents $\cos x$ for all x.

18. If $f(x) = \sin x$, then $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so by Formula 9 with $a=0$ and $M=1$, $|R_n(x)| \le \frac{1}{(n+1)!}\left|x-\frac{\pi}{2}\right|^{n+1}$. Thus, $|R_n(x)| \to 0$ as $n\to\infty$ by Equation 10. So $\lim_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 14 represents $\sin x$ for all x.

19. $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(\pi x) = \sum_{n=0}^{\infty}\frac{(-1)^n(\pi x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}x^{2n}}{(2n)!}$, $R=\infty$

20. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow f(x) = e^{-x/2} = \sum_{n=0}^{\infty}\frac{(-x/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\,n!}x^n$, $R=\infty$

21. $\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \Rightarrow f(x) = x\tan^{-1}x = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{2n+1}$, $R=1$

22. $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \Rightarrow f(x) = \sin(x^4) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{8n+4}$, $R=\infty$

23. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow f(x) = x^2e^{-x} = x^2\sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n!}$, $R=\infty$

24. $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \Rightarrow \cos 2x = \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}}{(2n)!}x^{2n} \Rightarrow$

$f(x) = x\cos 2x = \sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}}{(2n)!}x^{2n+1}$, $R=\infty$

25. $\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2}\left[1-\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\left[1-1-\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right]$

$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$, $R=\infty$

26. $\frac{x-\sin x}{x^3} = \frac{1}{x^3}\left[x-\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[x-x-\sum_{n=1}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[-\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\right]$

$= \frac{1}{x^3}\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+3}}{(2n+3)!} = \sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n+3)!}$

and this series also gives the required value at $x=0$ (namely 1/6); $R=\infty$.

27.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{1/2}$	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}(1+x)^{-7/2}$	$-\frac{15}{16}$
⋮	⋮	⋮

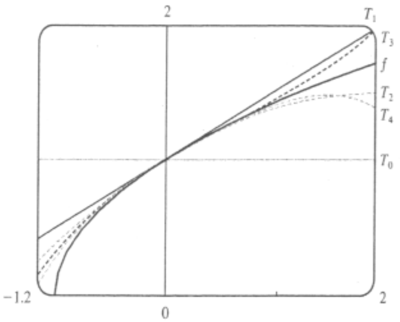
So $f^{(n)}(0) = \frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n}$ for $n\ge 2$, and $\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n$.

If $a_n = \frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)(2n-1)x^{n+1}}{2^{n+1}(n+1)!}\cdot\frac{2^n n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}\right|$$

$$= \frac{|x|}{2}\lim_{n\to\infty}\frac{2n-1}{n+1} = \frac{|x|}{2}\cdot 2 = |x| < 1 \text{ for convergence, so } R=1.$$

[continued]

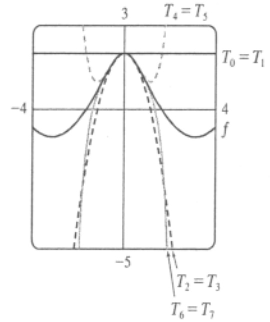


Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$ for $-1 < x < 1$.

28. $e^x \overset{(11)}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$.

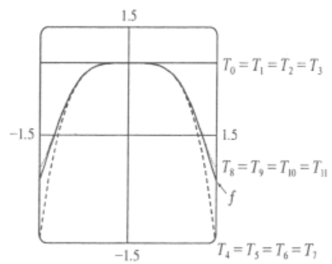
Also, $\cos x \overset{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, so

$$f(x) = e^{-x^2} + \cos x = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n!} + \frac{1}{(2n)!}\right) x^{2n}$$
$$= 2 - \frac{3}{2}x^2 + \frac{13}{24}x^4 - \frac{121}{720}x^6 + \cdots.$$



The series for e^x and $\cos x$ converge for all x, so the same is true of the series for $f(x)$; that is, $R = \infty$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.

29. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$, $R = \infty$

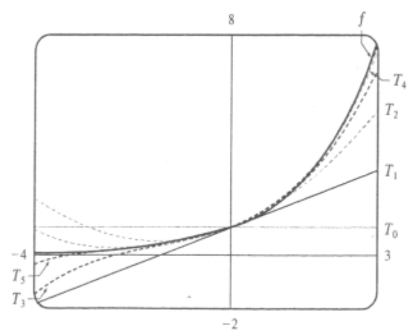


Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$.

30. $2^x = (e^{\ln 2})^x = e^{x \ln 2} = \sum_{n=0}^{\infty} \frac{(x \ln 2)^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{(\ln 2)^n x^n}{n!}$, $R = \infty$.

Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$.



31. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-0.2} = \sum_{n=0}^{\infty} \frac{(-0.2)^n}{n!} = 1 - 0.2 + \frac{1}{2!}(0.2)^2 - \frac{1}{3!}(0.2)^3 + \frac{1}{4!}(0.2)^4 - \frac{1}{5!}(0.2)^5 + \frac{1}{6!}(0.2)^6 - \cdots$.

But $\frac{1}{6!}(0.2)^6 = 8.\overline{8} \times 10^{-8}$, so by the Alternating Series Estimation Theorem, $e^{-0.2} \approx \sum_{n=0}^{5} \frac{(-0.2)^n}{n!} \approx 0.81873$, correct to five decimal places.

32. $3^\circ = \frac{\pi}{60}$ radians and $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, so

$\sin \frac{\pi}{60} = \frac{\pi}{60} - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \cdots = \frac{\pi}{60} - \frac{\pi^3}{1{,}296{,}000} + \frac{\pi^5}{93{,}312{,}000{,}000} - \cdots$. But $\frac{\pi^5}{93{,}312{,}000{,}000} < 10^{-8}$, so by the Alternating Series Estimation Theorem, $\sin \frac{\pi}{60} \approx \frac{\pi}{60} - \frac{\pi^3}{1{,}296{,}000} \approx 0.05234$.

33. $\cos x \overset{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} \Rightarrow$

$x\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \Rightarrow \int x\cos(x^3)\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, with $R = \infty$.

34. $\frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$, so $\int \frac{\sin x}{x}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$

35. Using the series from Exercise 27 and substituting x^3 for x, we get

$$\int \sqrt{x^3+1}\,dx = \int \left[1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^{3n}\right] dx$$
$$= C + x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n! (3n+1)} x^{3n+1}$$

36. $e^x \overset{(11)}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \Rightarrow \frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \Rightarrow \int \frac{e^x - 1}{x}\,dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$, with $R = \infty$.

37. By Exercise 33, $\int x\cos(x^3)\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, so

$\int_0^1 x\cos(x^3)\,dx = \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+2)(2n)!} = \frac{1}{2} - \frac{1}{8 \cdot 2!} + \frac{1}{14 \cdot 4!} - \frac{1}{20 \cdot 6!} + \cdots$, but $\frac{1}{20 \cdot 6!} = \frac{1}{14{,}400} \approx 0.000\,069$, so $\int_0^1 x\cos(x^3)\,dx \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{336} \approx 0.440$ (correct to three decimal places) by the Alternating Series Estimation Theorem.

38. From the table of Maclaurin series in this section, we see that

$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for x in $[-1, 1]$ and $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all real numbers x, so

$\tan^{-1}(x^3) + \sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$ for x^3 in $[-1, 1] \Leftrightarrow x$ in $[-1, 1]$. Thus,

$$I = \int_0^{0.2} [\tan^{-1}(x^3) + \sin(x^3)]\,dx = \int_0^{0.2} \sum_{n=0}^{\infty} (-1)^n x^{6n+3} \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right) dx$$
$$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{6n+4} \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right)\right]_0^{0.2} = \sum_{n=0}^{\infty} (-1)^n \frac{(0.2)^{6n+4}}{6n+4} \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right)$$
$$= \frac{(0.2)^4}{4}(1+1) - \frac{(0.2)^{10}}{10}\left(\frac{1}{3} + \frac{1}{3!}\right) + \cdots$$

[continued]

But $\dfrac{(0.2)^{10}}{10}\left(\dfrac{1}{3}+\dfrac{1}{3!}\right)=\dfrac{(0.2)^{10}}{20}=5.12\times10^{-9}$, so by the Alternating Series Estimation Theorem,

$I\approx\dfrac{(0.2)^4}{2}=0.000\,80$ (correct to five decimal places). [Actually, the value is 0.000 800 0, correct to seven decimal places.]

39. We first find a series representation for $f(x)=(1+x)^{-1/2}$, and then substitute.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-1/2}$	1
1	$-\frac{1}{2}(1+x)^{-3/2}$	$-\frac{1}{2}$
2	$\frac{3}{4}(1+x)^{-5/2}$	$\frac{3}{4}$
3	$-\frac{15}{8}(1+x)^{-7/2}$	$-\frac{15}{8}$
⋮	⋮	⋮

$$\frac{1}{\sqrt{1+x}}=1-\frac{x}{2}+\frac{3}{4}\left(\frac{x^2}{2!}\right)-\frac{15}{8}\left(\frac{x^3}{3!}\right)+\cdots\quad\Rightarrow\quad\frac{1}{\sqrt{1+x^3}}=1-\frac{1}{2}x^3+\frac{3}{8}x^6-\frac{5}{16}x^9+\cdots\quad\Rightarrow$$

$\displaystyle\int_0^{0.1}\frac{dx}{\sqrt{1+x^3}}=\left[x-\frac{1}{8}x^4+\frac{3}{56}x^7-\frac{1}{32}x^{10}+\cdots\right]_0^{0.1}\approx(0.1)-\frac{1}{8}(0.1)^4$, by the Alternating Series Estimation Theorem, since $\frac{3}{56}(0.1)^7\approx0.000\,000\,000\,5\,4<10^{-8}$, which is the maximum desired error. Therefore,

$$\int_0^{0.1}\frac{dx}{\sqrt{1+x^3}}\approx0.099\,987\,50.$$

40. $\displaystyle\int_0^{0.5}x^2e^{-x^2}\,dx=\int_0^{0.5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n!}\,dx=\sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+3}}{n!(2n+3)}\right]_0^{1/2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+3)2^{2n+3}}$ and since the term with $n=2$ is $\dfrac{1}{1792}<0.001$, we use $\displaystyle\sum_{n=0}^{1}\frac{(-1)^n}{n!(2n+3)2^{2n+3}}=\frac{1}{24}-\frac{1}{160}\approx0.0354$.

41.
$$\lim_{x\to0}\frac{x-\tan^{-1}x}{x^3}=\lim_{x\to0}\frac{x-\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots\right)}{x^3}=\lim_{x\to0}\frac{\frac{1}{3}x^3-\frac{1}{5}x^5+\frac{1}{7}x^7-\cdots}{x^3}$$
$$=\lim_{x\to0}\left(\tfrac{1}{3}-\tfrac{1}{5}x^2+\tfrac{1}{7}x^4-\cdots\right)=\tfrac{1}{3}$$

since power series are continuous functions.

42.
$$\lim_{x\to0}\frac{1-\cos x}{1+x-e^x}=\lim_{x\to0}\frac{1-\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)}{1+x-\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\cdots\right)}$$
$$=\lim_{x\to0}\frac{\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6-\cdots}{-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5-\frac{1}{6!}x^6-\cdots}$$
$$=\lim_{x\to0}\frac{\frac{1}{2!}-\frac{1}{4!}x^2+\frac{1}{6!}x^4-\cdots}{-\frac{1}{2!}-\frac{1}{3!}x-\frac{1}{4!}x^2-\frac{1}{5!}x^3-\frac{1}{6!}x^4-\cdots}=\frac{\frac{1}{2}-0}{-\frac{1}{2}-0}=-1$$

since power series are continuous functions.

43.
$$\lim_{x\to0}\frac{\sin x-x+\frac{1}{6}x^3}{x^5}=\lim_{x\to0}\frac{\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots\right)-x+\frac{1}{6}x^3}{x^5}$$
$$=\lim_{x\to0}\frac{\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots}{x^5}=\lim_{x\to0}\left(\frac{1}{5!}-\frac{x^2}{7!}+\frac{x^4}{9!}-\cdots\right)=\frac{1}{5!}=\frac{1}{120}$$

since power series are continuous functions.

44.
$$\lim_{x\to0}\frac{\tan x-x}{x^3}=\lim_{x\to0}\frac{\left(x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots\right)-x}{x^3}=\lim_{x\to0}\frac{\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots}{x^3}=\lim_{x\to0}\left(\tfrac{1}{3}+\tfrac{2}{15}x^2+\cdots\right)=\tfrac{1}{3}$$

since power series are continuous functions.

45. As in Example 8(a), we have $e^{-x^2}=1-\dfrac{x^2}{1!}+\dfrac{x^4}{2!}-\dfrac{x^6}{3!}+\cdots$ and we know that $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\cdots$ from Equation 16. Therefore, $e^{-x^2}\cos x=\left(1-x^2+\frac{1}{2}x^4-\cdots\right)\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots\right)$. Writing only the terms with degree ≤4, we get $e^{-x^2}\cos x=1-\frac{1}{2}x^2+\frac{1}{24}x^4-x^2+\frac{1}{2}x^4+\frac{1}{2}x^4+\cdots=1-\frac{3}{2}x^2+\frac{25}{24}x^4+\cdots$.

46. $\sec x=\dfrac{1}{\cos x}\overset{(16)}{=}\dfrac{1}{1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots}$.

$$\begin{array}{r|l}
 & 1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots \\
1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots & 1 \\
 & 1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots \\
 & \frac{1}{2}x^2-\frac{1}{24}x^4+\cdots \\
 & \frac{1}{2}x^2-\frac{1}{4}x^4+\cdots \\
 & \frac{5}{24}x^4+\cdots \\
 & \frac{5}{24}x^4+\cdots \\
 & \cdots
\end{array}$$

From the long division above, $\sec x=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots$.

47. $\dfrac{x}{\sin x}\overset{(15)}{=}\dfrac{x}{x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots}$.

$$\begin{array}{r|l}
 & 1+\frac{1}{6}x^2+\frac{7}{360}x^4+\cdots \\
x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots & x \\
 & x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots \\
 & \frac{1}{6}x^3-\frac{1}{120}x^5+\cdots \\
 & \frac{1}{6}x^3-\frac{1}{36}x^5+\cdots \\
 & \frac{7}{360}x^5+\cdots \\
 & \frac{7}{360}x^5+\cdots \\
 & \cdots
\end{array}$$

From the long division above, $\dfrac{x}{\sin x}=1+\frac{1}{6}x^2+\frac{7}{360}x^4+\cdots$.

48. From Example 6 in Section 8.6, we have $\ln(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots$, $|x|<1$. Therefore,

$e^x\ln(1-x)=\left(1+x+\frac{1}{2}x^2+\cdots\right)\left(-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots\right)$

$=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-x^2-\frac{1}{2}x^3-\frac{1}{2}x^3-\cdots=-x-\frac{3}{2}x^2-\frac{4}{3}x^3-\cdots$, $|x|<1$

49. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!}=\sum_{n=0}^{\infty}\frac{\left(-x^4\right)^n}{n!}=e^{-x^4}$, by (11).

50. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, by (16).

51. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, by (15).

52. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{5^n\,n!}=\sum_{n=0}^{\infty}\frac{(3/5)^n}{n!}=e^{3/5}$, by (11).

53. $3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots=\frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots=\sum_{n=1}^{\infty}\frac{3^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n}{n!}-1=e^3-1$, by (11).

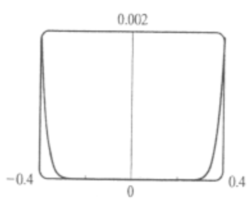
54. $1-\ln 2+\frac{(\ln 2)^2}{2!}-\frac{(\ln 2)^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{(-\ln 2)^n}{n!}=e^{-\ln 2}=(e^{\ln 2})^{-1}=2^{-1}=\frac{1}{2}$, by (11).

55. Assume that $|f'''(x)|\le M$, so $f'''(x)\le M$ for $a\le x\le a+d$. Now $\int_a^x f'''(t)\,dt\le\int_a^x M\,dt \Rightarrow$
$f''(x)-f''(a)\le M(x-a) \Rightarrow f''(x)\le f''(a)+M(x-a)$. Thus, $\int_a^x f''(t)\,dt\le\int_a^x[f''(a)+M(t-a)]\,dt \Rightarrow$
$f'(x)-f'(a)\le f''(a)(x-a)+\frac{1}{2}M(x-a)^2 \Rightarrow f'(x)\le f'(a)+f''(a)(x-a)+\frac{1}{2}M(x-a)^2 \Rightarrow$
$\int_a^x f'(t)\,dt\le\int_a^x\left[f'(a)+f''(a)(t-a)+\frac{1}{2}M(t-a)^2\right]dt \Rightarrow$
$f(x)-f(a)\le f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\frac{1}{6}M(x-a)^3$. So
$f(x)-f(a)-f'(a)(x-a)-\frac{1}{2}f''(a)(x-a)^2\le\frac{1}{6}M(x-a)^3$. But
$R_2(x)=f(x)-T_2(x)=f(x)-f(a)-f'(a)(x-a)-\frac{1}{2}f''(a)(x-a)^2$, so $R_2(x)\le\frac{1}{6}M(x-a)^3$.
A similar argument using $f'''(x)\ge -M$ shows that $R_2(x)\ge-\frac{1}{6}M(x-a)^3$. So $|R_2(x_2)|\le\frac{1}{6}M\,|x-a|^3$.
Although we have assumed that $x>a$, a similar calculation shows that this inequality is also true if $x<a$.

56. (a) $f(x)=\begin{cases}e^{-1/x^2} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ so $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{e^{-1/x^2}}{x}=\lim_{x\to 0}\frac{1/x}{e^{1/x^2}}=\lim_{x\to 0}\frac{x}{2e^{1/x^2}}=0$

(using l'Hospital's Rule and simplifying in the penultimate step). Similarly, we can use the definition of the derivative and l'Hospital's Rule to show that $f''(0)=0, f^{(3)}(0)=0,\ldots,f^{(n)}(0)=0$, so that the Maclaurin series for f consists entirely of zero terms. But since $f(x)\ne 0$ except for $x=0$, we see that f cannot equal its Maclaurin series except at $x=0$.

(b)



From the graph, it seems that the function is extremely flat at the origin. In fact, it could be said to be "infinitely flat" at $x=0$, since all of its derivatives are 0 there.

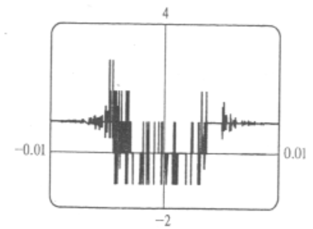
LABORATORY PROJECT An Elusive Limit

1. $f(x)=\frac{n(x)}{d(x)}=\frac{\sin(\tan x)-\tan(\sin x)}{\arcsin(\arctan x)-\arctan(\arcsin x)}$

x	$f(x)$
1	1.1838
0.1	0.9821
0.01	2.0000
0.001	3.3333
0.0001	3.3333

The table of function values were obtained using Maple with 10 digits of precision. The results of this project will vary depending on the CAS and precision level. It appears that as $x\to 0^+$, $f(x)\to\frac{10}{3}$. Since f is an even function, we have $f(x)\to\frac{10}{3}$ as $x\to 0$.

2. The graph is inconclusive about the limit of f as $x\to 0$.



3. The limit has the indeterminate form $\frac{0}{0}$. Applying l'Hospital's Rule, we obtain the form $\frac{0}{0}$ six times. Finally, on the seventh application we obtain $\lim_{x\to 0}\frac{n^{(7)}(x)}{d^{(7)}(x)}=\frac{-168}{-168}=1$.

4. $\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{n(x)}{d(x)}\overset{\text{CAS}}{=}\lim_{x\to 0}\frac{-\frac{1}{30}x^7-\frac{29}{756}x^9+\cdots}{-\frac{1}{30}x^7+\frac{13}{756}x^9+\cdots}$

$$=\lim_{x\to 0}\frac{\left(-\frac{1}{30}x^7-\frac{29}{756}x^9+\cdots\right)/x^7}{\left(-\frac{1}{30}x^7+\frac{13}{756}x^9+\cdots\right)/x^7}=\lim_{x\to 0}\frac{-\frac{1}{30}-\frac{29}{756}x^2+\cdots}{-\frac{1}{30}+\frac{13}{756}x^2+\cdots}=\frac{-\frac{1}{30}}{-\frac{1}{30}}=1$$

Note that $n^{(7)}(x)=d^{(7)}(x)=-\frac{7!}{30}=-\frac{5040}{30}=-168$, which agrees with the result in Problem 3.

5. The limit command gives the result that $\lim_{x\to 0}f(x)=1$.

6. The strange results (with only 10 digits of precision) must be due to the fact that the terms being subtracted in the numerator and denominator are very close in value when $|x|$ is small. Thus, the differences are imprecise (have few correct digits).

8.8 The Binomial Series

1. The general binomial series in (2) is

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots.$$

$$(1+x)^{1/2}=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n=1+\left(\tfrac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+\cdots$$

$$=1+\frac{x}{2}-\frac{x^2}{2^2\cdot 2!}+\frac{1\cdot 3\cdot x^3}{2^3\cdot 3!}-\frac{1\cdot 3\cdot 5\cdot x^4}{2^4\cdot 4!}+\cdots$$

$$=1+\frac{x}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}{2^n\cdot n!}\text{ for }|x|<1,\quad\text{so }R=1.$$

2. $\frac{1}{(1+x)^4}=(1+x)^{-4}=\sum_{n=0}^{\infty}\binom{-4}{n}x^n$. The binomial coefficient is

$$\binom{-4}{n}=\frac{(-4)(-5)(-6)\cdot\cdots\cdot(-4-n+1)}{n!}=\frac{(-4)(-5)(-6)\cdot\cdots\cdot[-(n+3)]}{n!}$$

$$=\frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot\cdots\cdot(n+1)(n+2)(n+3)}{2\cdot 3\cdot n!}=\frac{(-1)^n(n+1)(n+2)(n+3)}{6}$$

Thus, $\frac{1}{(1+x)^4}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)(n+3)}{6}x^n$ for $|x|<1$, so $R=1$.