

MTH U242, Practice Final'08

1. Evaluate the integrals:

(a)  $\int \frac{\sin 2x}{1+\cos^2 x} dx$ , **Answer:**  $-\ln(1 + \cos^2 x) + c$

(b)  $\int \sin x \ln(\cos x) dx$ , **Answer:**  $-\cos x \ln |\cos x| + \cos x + C$

(c)  $\int \frac{dx}{\sqrt{1-x^2}}$ , **Answer:**  $\arcsin x + C$

(d)  $\int \frac{x^3}{\sqrt{1-x^4}} dx$ , **Answer:**  $-\frac{1}{2}\sqrt{1-x^4} + C$

(e)  $\int xe^x dx$ , **Answer:**  $xe^x - e^x + C$

(f)  $\int \frac{2x+1}{(x+1)(x+2)} dx$ , **Answer:**  $3 \ln |x+2| - \ln |x+1| + C$

2. Determine whether the following improper integrals are convergent:

(a)  $\int_0^\infty e^{-x} dx$ , **Answer:** 1, converges

(b)  $\int_0^1 \ln x dx$ , **Answer:** -1, converges

(c)  $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$ , **Answer:**  $\frac{32}{3}$ , converges

3. Find the area of the region enclosed by

(a) the line  $y = x - 1$  and the parabola  $y^2 = 2x + 6$ , **Answer:** 18

(b)  $x = 2y^2$  and  $x + y = 1$ , **Answer:**  $\frac{9}{8}$

4(a) Find the volume of the solid you get if the region enclosed by the parabolas  $y = x^2$ ,  $y^2 = 8x$ , is rotated about the  $x$ -axis. **Answer:**  $\frac{48}{5}\pi$

4(b) The region enclosed by the curves  $y = x^3$  and  $y = \sqrt{x}$  is rotated about the line  $x = 1$ . Find the volume of the resulting solid. **Answer:**  $\frac{13}{30}\pi$

5(a) A cable that weighs 2lb/ft is used to lift 800 lb coal up a mineshaft 500 ft deep. Find the work done. **Answer:**  $65 \times 10^4$  [ft-lb]

5(b) The vertical end of a tank is an isosceles triangle with height 3 m and base 3 m. Assume the tank is 8 m long and that it is full of water and that the water is to be pumped to a height 2 m above the top of the tank. Find the work done in emptying the tank. **Answer:**  $\approx 1.0584 \times 10^6$  [J]

6. Find the limits of the sequences:

(a)  $\lim_{n \rightarrow \infty} \frac{n^4+7}{7n^4} = \frac{1}{7}$

(b)  $\lim_{n \rightarrow \infty} \frac{(\ln(n))^2}{n^3} = 0$  (l'Hôpital Rule)

(c)  $\lim_{n \rightarrow \infty} \frac{\sqrt{n}+1}{n} = 0.$

7. If the following series converges, give a number to which it converges:

(a)  $\sum_{n=0}^{\infty} \frac{253}{(-3)^n} = \frac{759}{4}$  (geometric series)

(b)  $\sum_{n=0}^{\infty} \frac{2^n+1}{3^n} = \frac{9}{2}$  (the sum of two geometric series)

(c)  $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = \frac{1}{3}$  (collapsing series)

8. Determine whether the following series are convergent or divergent:

(a)  $\sum_{n=0}^{\infty} \frac{n}{n+1}$  **Answer:** divergent by the Divergence Test

(b)  $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$  **Answer:** convergent by the Alternating Series Test

(c)  $\sum_{n=1}^{\infty} \frac{\sin^2 n}{\sqrt{n^3}}$  **Answer:** convergent by the Comparison Test

(d)  $\sum_{n=1}^{\infty} (-1)^n \frac{2}{n+5}$  **Answer:** convergent by the Alternating Series Test

9. Determine the interval of convergence and the radius of convergence of the power series:

(a)  $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)3^n}$  **Answer:**  $-3 \leq x < 3$ ,  $R = 3$

(b)  $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$  **Answer:**  $0 \leq x < 2$ ,  $R = 1$

10. Let  $f(x) = \frac{x^2}{4+x^2}$ ,  $-2 < x < 2$ .

(a) Find a power series representation for  $f$

$$\text{Answer: } \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^{n+1}}$$

(b) Find a power series representation for  $\int \frac{x^2}{4+x^2} dx$

$$\text{Answer: } C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+3)4^{n+1}}$$

(c) Find a power series representation for  $f'$

$$\text{Answer: } \sum_{n=0}^{\infty} (-1)^n (2n+2) \frac{x^{2n+1}}{4^{n+1}}$$

11. Find the Taylor series of  $f(x)$  at  $a$

(a)  $f(x) = \cos x$ ,  $a = \pi$     **Answer:**  $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!}$

(b)  $f(x) = e^{3x}$ ,  $a = 0$     **Answer:**  $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

(c)  $f(x) = \sin x$ ,  $a = \pi/2$     **Answer:**  $\sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}$

12. Consider the vectors  $\vec{a} = \langle 1, 2, 3 \rangle$  and  $\vec{b} = \langle -1, -2, 3 \rangle$ .

(a) Find  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$ .

$$\text{Answer: } \vec{a} + \vec{b} = \langle 0, 0, 6 \rangle, \vec{a} - \vec{b} = \langle 2, 4, 0 \rangle$$

(b) Find the dot product  $\vec{a} \cdot \vec{b}$ .

$$\text{Answer: } \vec{a} \cdot \vec{b} = 4$$

(c) Find the angle between  $\vec{a}$  and  $\vec{b}$ .

$$\text{Answer: } \approx 1.28104$$

13. Find a vector equation and parametric equations for the line connecting the points  $P(1, -1, 2)$  and  $Q(5, 2, 8)$ .

**Answer:** Vector equation:

$$\vec{r}(t) = \langle 1 + 4t, -1 + 3t, 2 + 6t \rangle, \quad 0 \leq t \leq 1$$

Parametric equations:

$$\{x = 1 + 4t, y = -1 + 3t, z = 2 + 6t\}$$

14. Let  $P(4, 3, 5)$ ,  $Q(1, 1, 1)$  and  $R(-1, 10, -2)$  be the vertices of the triangle  $\triangle PQR$ .

- (a) Find the angle at  $P$ ; **Answer:**  $\approx 1.06379$  [rad]  
(b) Is it true that the angle at  $Q$  is right? If "Yes", what is the length of the hypotenuse? **Answer:** Yes, the angle is right,  $|PR| = \sqrt{123}$

15. Consider the vector function  $\vec{r}(t) = \langle 1, t^2, t^3 \rangle$ ,  $0 \leq t \leq 1$ .

- (a) Find the derivative  $\vec{r}'(t)$  **Answer:**  $\vec{r}'(t) = \langle 0, 2t, 3t^2 \rangle$   
(b) Find the length of the curve. **Answer:**  $L = \frac{1}{27}(13^{3/2} - 8)$

16. Find the unit tangent and normal vector to the curve  $\vec{r}(t) = \langle \sin 3t, \cos 3t, t \rangle$  at  $t = \pi/9$ .

**Answer:**

$$\vec{T}(\pi/9) = \left\langle \frac{3}{2\sqrt{10}}, -\frac{3\sqrt{3}}{2\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$$

$$\vec{N}(\pi/9) = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0 \right\rangle$$