The Logistic Differential Equation

A more realistic model for population growth in most circumstances, than the exponential model, is provided by the Logistic Differential Equation. In this case one's assumptions about the growth of the population include a maximum size beyond which the population cannot expand. This may be due to a space limitation, a ceiling on the food supply or the number of people concerned in the case of the spread of a rumor.

One then assumes that the growth rate of the population is proportional to number of individuals present, P(t), but that this rate is now constrained by how close the number P(t) is to the maximum size possible for the population, M. A natural way to include this assumption in your mathematical model is to posit that

$$P'(t) = k \cdot P(t) \cdot \left(M - P(t)\right)$$

This is the Logistic Differential Equation. We will now solve this equation.

We separate the variables in the equation: $\dfrac{dP}{dt} = kP(M - P)$

to obtain $\dfrac{dP}{P(M - P)} = kdt$

Use partial fractions on the LHS to get: $\dfrac{dP}{P(M - P)} = \left(\dfrac{\frac{1}{m}}{P} + \dfrac{\frac{1}{m}}{M - P}\right)dP$

Integrating both sides of the equation now yields

$$\frac{1}{M}\ln P - \frac{1}{m}\ln(M - P) = kt + c$$

or, $\ln\left(\dfrac{P}{M - P}\right) = Mkt + c$, so that $\dfrac{P}{M - P} = e^{Mkt + c}$

Then P = (M – P)$\cdot e^{Mkt + c}$ so

$$P + P \cdot e^{Mkt + c} = M \cdot e^{Mkt + c}$$

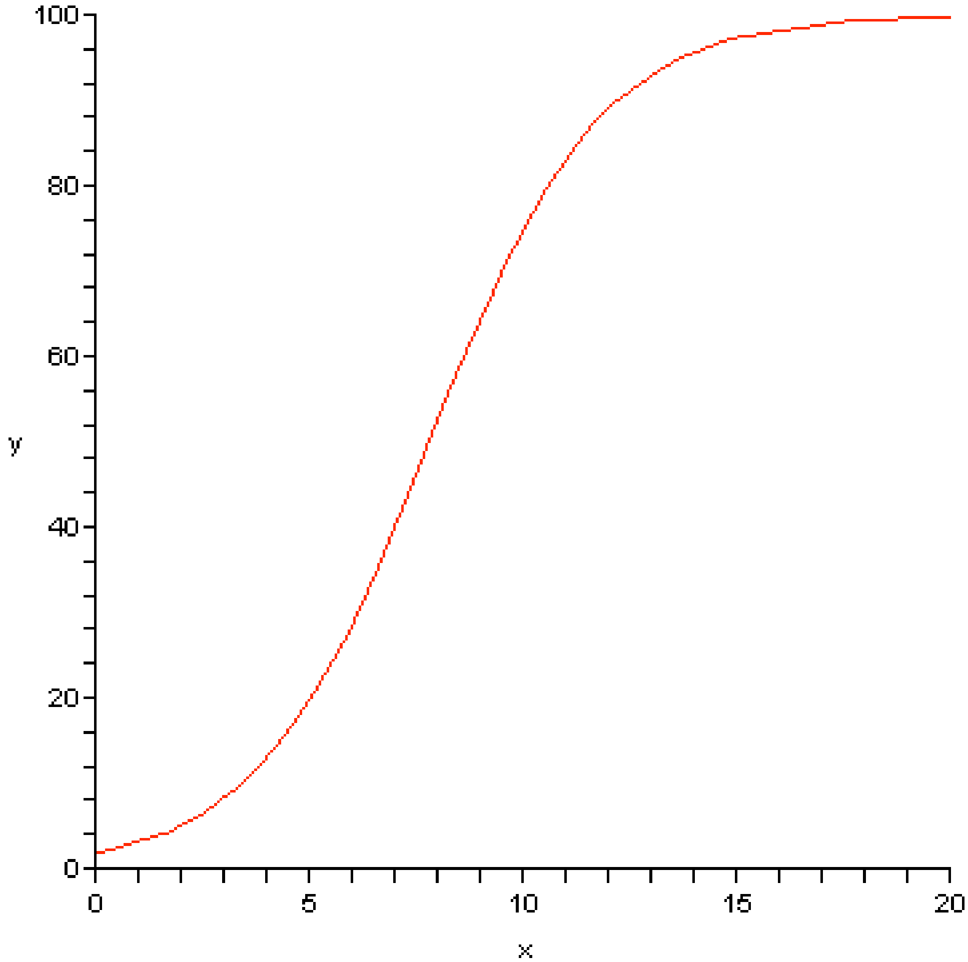
$$P = \frac{M \cdot e^{Mkt + c}}{1 + e^{Mkt + c}} = \frac{M}{1 + A \cdot e^{-Mkt}}, \text{ where } A = e^{-c}$$

Now set t = 0: $P_0 = \dfrac{M}{1 + A}$, $P_0 \cdot (1 + A) = M$, $A = \dfrac{M - P_0}{P_0}$

So A is the ratio of the room that the population has to grow when t = 0, to the size of P(t) when t = 0. Since $P(t) = \dfrac{M}{1 + Ae^{-Mkt}}$, it is clear that $as\ t \to \infty,\ P(t) \to M.$

Also, when t = 0, the equation for P gives P(t) = P_0.

A graph with M = 100, P(0) = 2 and k = .005 is below. It has the shape of the "sigma" curve associated with logistic growth.



100
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