

Example: Group action of $G = D_4$ on the set $\text{Vert} = \{A, B, C, D\}$.

Stabilizer subgroup $\text{Stab}_G A = (e, (BD))$ (here BD is the flip about axis AC).

Orbit of A under D_4 : =subset of all vertices hit by action of D_4 on A = all Vertices (for each vertex, there is a rotation taking A to it).

Formula

$$\#G = (\# \text{Stab}_A)(\# \text{Orb of } A)$$

Here we have $\#D_4 = 2(4) = 8$.

Application to size of Dihedral group: show $\#D_5 = 2(5)$, and $\#D_n = 2n$.

Theorem (stated without proof): Let G act on a set S , let $H = \text{Stab}_a$, the subgroup of G fixing a . Let g_1, \dots, g_s be elements of G , such that

$$g_1 \circ a, \dots, g_s \circ a = \text{Orb}(A) \quad (\text{with no repetitions}).$$

Then

$$G = \{g_1, \dots, g_s\} H$$

and this is a direct product of sets.

In other words, each element of G can be written uniquely as a product of an element from $\{g_1, \dots, g_s\}$ (on the left) times an element of H .

[This is shown during the proof of Theorem 7.3 in text]

As it is a rich source of information on symmetry groups and their subgroups, we will apply it now.

Application to finding the symmetry group G of infinite train tracks with equally spaced ties (Worksheet 1, #Ae).

A. We used the vertex $A = "0"$, the set S of all vertices (intersections of ties with rails). $\text{Stab}_G A = \{e, f_v\}$. $f_v = f$; flip about the tie containing A .

Orbit of A is all vertices.

Exercise 1: Apply this to determine G as a set of symmetries.

B. We next took the set T of midpoints of ties. For M a midpoint, $\text{Stab}_G M = \{e, f_v, f_h, \text{rot } 180 \text{ deg}\} = W$, the symmetry group of a rectangle (formed by the box of two ties closest to but not containing M , with the rails).

Orbit(M) = all midpoints. The translations take M to the other midpoints.

Exercise 2: Apply this to write each element of the symmetry group of the tracks, as a product of a translation times an element of W .

Exercise 3; Apply to determining the symmetry group of a circle. (WS1Bd).

Web References: [Rowland, Todd](#) and [Weisstein, Eric W.](#) "Group Orbit." From [MathWorld](#)--A Wolfram Web Resource. <http://mathworld.wolfram.com/GroupOrbit.html>

<http://www.sju.edu/~pklingsb/burnside.pdf> (Dr. Paul Klingsberg: St Joseph University)