

**Example:** Group action of  $G = D_4$  on the set  $\text{Vert} = \{A, B, C, D\}$ .

Stabilizer subgroup  $\text{Stab}_G A = (e, (BD))$  (here  $BD$  is the flip about axis  $AC$ ).

Orbit of  $A$  under  $D_4$ : =subset of all vertices hit by action of  $D_4$  on  $A$  = all Vertices (for each vertex, there is a rotation taking  $A$  to it).

Formula

$$\#G = (\# \text{Stab}_A)(\# \text{Orb of } A)$$

Here we have  $\#D_4 = 2(4) = 8$ .

Application to size of Dihedral group: show  $\#D_5 = 2(5)$ , and  $\#D_n = 2n$ .

**Theorem** (stated without proof): Let  $G$  act on a set  $S$ , let  $H = \text{Stab}_a$ , the subgroup of  $G$  fixing  $a$ . Let  $g_1, \dots, g_s$  be elements of  $G$ , such that

$$g_1 \circ a, \dots, g_s \circ a = \text{Orb}(A) \quad (\text{with no repetitions}).$$

Then

$$G = \{g_1, \dots, g_s\} H$$

and this is a direct product of sets.

In other words, each element of  $G$  can be written uniquely as a product of an element from  $\{g_1, \dots, g_s\}$  (on the left) times an element of  $H$ .

[This is shown during the proof of Theorem 7.3 in text]

As it is a rich source of information on symmetry groups and their subgroups, we will apply it now.

**Application** to finding the symmetry group  $G$  of infinite train tracks with equally spaced ties (Worksheet 1, #Ae).

A. We used the vertex  $A = "0"$ , the set  $S$  of all vertices (intersections of ties with rails).  $\text{Stab}_G A = \{e, f_v\}$ .  $f_v = f$ ; flip about the tie containing  $A$ .

Orbit of  $A$  is all vertices.

**Exercise 1:** Apply this to determine  $G$  as a set of symmetries.

B. We next took the set  $T$  of midpoints of ties. For  $M$  a midpoint,  $\text{Stab}_G M = \{e, f_v, f_h, \text{rot } 180 \text{ deg}\} = W$ , the symmetry group of a rectangle (formed by the box of two ties closest to but not containing  $M$ , with the rails).

Orbit( $M$ ) = all midpoints. The translations take  $M$  to the other midpoints.

**Exercise 2:** Apply this to write each element of the symmetry group of the tracks, as a product of a translation times an element of  $W$ .

**Exercise 3;** Apply to determining the symmetry group of a circle. (WS1Bd).

Web References: [Rowland, Todd](#) and [Weisstein, Eric W.](#) "Group Orbit." From [MathWorld](#)--A Wolfram Web Resource. <http://mathworld.wolfram.com/GroupOrbit.html>

<http://www.sju.edu/~pklingsb/burnside.pdf> (Dr. Paul Klingsberg: St Joseph University)