Math U575 Spring 08 Half Quiz 3.5 Prof. A. Iarrobino Name:___________________
Please show your work for full credit. The best 2 out of 3 problems will be counted.

1. Let g, h be disjoint p-cycles in the symmetric group S_{2p}, p an odd prime. Let H=<g,h> be the subgroup of S_{2p} generated by g and h. Show that H is isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$. You may assume that the map of sets

$$\varphi: \mathbf{Z}_p \times \mathbf{Z}_p \to H,\ \varphi(a,b) = g^a h^b \quad \text{is well-defined.}$$

Steps: Show that φ is also
i. 1-1 and onto, *and*
ii. a homomorphism.

1b. i. Where in the proof do you need the cycles g,h to be disjoint?

ii*. (EC) Discuss the subgroup lattice of H (how many subgroups, what are the inclusions?).

2. Let w=(a,b,c,d,e), v=(d,e,f,g,h) be two 5-cycles in S_8 (permutations of {a,b,…,h}) having two entries in common. What can you say about the subgroup W of S_8 they generate? Can you determine the shapes of the possible disjoint cycle decompositions for elements of W? Some possible steps are given in i-iii below.
Recall that by shape is meant the partition of 8 given by the lengths of the disjoint cycles in a decomposition. For example, the shape of (a,b,c)(d,e)(f,g)(h) is (3,2,2,1).
i. Determine the elements wv, wvw^{-1} the commutator C=$wvw^{-1}v^{-1}$ of w and v, and as well wC.
ii. For each such element, determine the order of the cyclic group it generates, and also the shapes of the cycle decompositions for the generating element and for its powers.
iii. Can (5,2,1), a 5-cycle times a transposition, occur for an element of W? Why or why not?

3a. Consider the map $\theta : S_3 \rightarrow S_6 : \theta(g) = \textit{left multiplication by } g \textit{ on the elements of } S_3$.
We term the analogous map for a group G the *Cayley map:* $G \rightarrow S_n,\ n = |S_n|$.
Label the elements of S_3

$a = (1,2),\ b = (1,3),\ c = (2,3),\ d = (1,2,3),\ e = identity,\ f = (1,3,2)$.

Note that $\theta(1,2)(a) = (1,2)(1,2) = e$, and $\theta(1,2)(d) = (1,2)(1,2,3) = c$
Using this labeling determine the disjoint cycle decompositions of
i. $\theta(1,2,3)$.
ii. $\theta(1,2)$

3b. Let: $\theta : G \rightarrow S_n, n = |G|$ be the Cayley map,. That is, $\theta(g) \in S_n$ is the permutation of the elements of G given by left multiplication by g. as above for S_3.
Show that $\theta(g^{-1}) = (\theta(g))^{-1}$, by showing first that $\theta(g^{-1}) \circ \theta(g)\ \ (h) = h$, for all h in S_3. Explain why this shows that $\theta(g^{-1})$ is the inverse of $\theta(g)$ (Please do not assume that θ is a homomophism).

3c (Extra Credit) Discuss the connection between right cosets of a cyclic subgroup H=<g> in G, and the cycle decomposition of $\theta(g)$, Recall that a right coset has the form Ha for some element $a \in G$

i. Show that each disjoint cycle of $\theta(g)$ is a permutation of some right coset of H.

ii. Use (i.) to determine the shape of the disjoint cycle decomposition of $\theta(g) \in S_n$ in terms of the order |H|.

(Recall that the shape of a disjoint cycle decomposition of an element σ of S_n is the partition of n given by the lengths of the disjoint cycles of σ. See #2 for an example