Math U575: Notes for Exam 1, Prof. A. Iarrobino section.
Hour Exam: Thursday March 20. Chapters 1-9 (up to normal subgroups)

The exam will test basic knowledge of the course to date. Most questions quite close to some of those on past quizzes, but on average shorter. I would like well prepared students to finish the exam and be able to have a few minutes to look over your work.

I plan 6 questions, scored best 5 of 6 at 20 points each, and up to two extra credit questions (10 pts each).

Here are some sample questions: Most are from quizzes 2,3,and 4.

1. Consider the subgroup H = (15) of Z_{20}. Determine
   a. the order of H
   b. list the subgroups of H
   c. determine the generators of H
   d. H is isomorphic to what cyclic group?

2. Using the definition of group, prove that in each row of the Cayley table, each element of the group occurs exactly once. You will need to choose g in G, consider the g row, and show i. left cancellation, *gx=gy implies x=y, and also show that*
   ii. Let b be an element of G. Show there is an element x such that ax=b.
   Be sure to clearly indicate a reason for each step of your proof.

3.a. Determine the two non-isomorphic groups of order 14.
b. Assuming that each group of order 14 contains a cyclic subgroup of order 7, show that your list is complete.

4a. Describe 4 distinct groups of order 8 (there are five).
4b. Show that if every element of a group G has order 2 or 1, then G is Abelian.
4c. Also show that a finite group G as in (4b) is a direct sum of copies of Z_2.

5. Let H be a subgroup of G. Show that if the left coset aH intersects the left coset bH, then the cosets are equal. (*This underlies the result about cosets partitioning a group, so please do not assume the partitioning result).*

6. Consider the rotation group G of the regular tetrahedron T with vertices A,B,C,D, as a subgroup of the permutation group S_4 of the set {A,B,C,D).
i. Let G act on the set F of faces {ABC,ABD,ACD,BCD} of T. Determine the isotropy group H =of the face $F_1 = ABD$, and also determine the orbit of F_1.
ii. Interpret the left coset (ABC)H geometrically.
iii. *Explain why (ABC)H(ACB) is the isotropy group of a different face, and identify that face.*
iv. Define ``alternating group $A_4 \subset S_4$, and show that G is isomorphic to A_4.

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7. Consider the group W= $S_4 \oplus S_3 \subset S_7$ Here S_4 is the group of permutations of {A,B,C,D}, S_3 the group of permutations of {E,F,G}, and S_7 the group or permutations of (A,B,....,G). Determine the following about W. Explain your answers

a. The order of W.

b. The maximum order of a cyclic subgroup of W.

c. Whether the subgroup S_4 is a normal subgroup of W.

d. The product gh in W, where g=((ABC)(D),(EF)(G)), and h=((A)(BCD),(E)(FG)),

e. The order of g in W.

f. The index of W in S_7.

Note: in the following two problems concerning isomorphism, please take care to

i. define a map from one group to another,

ii. show that it is a homomorphism,

iii. and show the map is 1-1 (injective) and onto (surjective).

8. Show that the map τ : **C**, + to **C**,+ $\tau(a+bi) = a - bi$ is an isomorphism of *additive* groups. You need to show τ is 1-1, onto, and satisfies a homomorphism property.

9. . Let g, h be disjoint 3-cycles in the symmetric group S_6. Show that the subgroup H=<g,h> of S_6 generated by g and h is isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3$.

10. The dihedral group D_6 acting on the regular hexagon with vertices A,B,C,D,E,F acts also on the set P ={(AB,DE),(BC,EF), (DC,FA)}= (P_1, P_2, P_3) of three pairs of opposite sides. The dihedral group is comprised of rotations, generated by r = (ABCDEF), three flips (AB)(DE), (BC)(EF),(CD,EF) about the centers of sides, and three flips (AC,DF),(BD)(AE), (CE)(BF) about pairs of opposite vertices.

a. Determine the isotropy group H of $P_1 = (AB, DE)$.

b. Determine the orbit of P_1.

c. Explain geometrically the left coset r^2H . Show that r^2Hr^4 is the isotropy group of a different pair, and determine which.

Extra Credit:

A. Describe the 5 non-isomorphic groups of order 12 ,and show that they are (pairwise) non-isomorphic.

B. Let S_7 be the symmetric group on 7 elements {A,B,C,D,E,F,G}. Let g,h be 5-cycles having 3 elements in common: g= (ABCDE) h=(CDEFG). Discuss the subgroup H=<g,h> generated by g and h (all products involving g,h, and their inverses). Determine the possible orders of elements of H. Decide if $H \subset A_7$ and can you determine the order of H?

C. Outline a proof that a group of order 2p, p a prime is either Z_{2p} or the dihedral group D_p of order p.