What is a Mathematical Model?
Uses and Limitations - A. Iarrobino for MTH 1108

1. What is a mathematical model?

A mathematical model is an equation which is intended to match or model the behavior of some natural quantities. Often it is a differential equation involving the rate of change **y'** with respect to time, and some function f(y,t):

$$y' = f(y,t). \qquad (1)$$

Then there will be a <u>solution</u> of the differential equation,

$$y = y(t,c) \qquad (2)$$

involving a constant determined by the initial conditions.

Examples:

A. ***Exponential growth/decay*:** **y' = ky**, with <u>solution</u> $y = Ae^{kt}$.
Here k is determined by the physical properties of the substance while A is the initial amount at time zero of the substance.

B. ***Newton's law of cooling*** : **y' = k(r-y)**, with <u>solution</u> $y = r+Ae^{-kt}$.
Here r is the limit temperature and is an *equilibrium* value of y, [since $y(t_0) = r$ implies $y'(t_0) = 0$, so y stays constant, y(t) = r].
The constant k is determined by the physical situation (the glass, the milk, air, conductance of the table, if a glass of milk is taken from a refrigerator and place on the kitchen table). Here A is determined by the initial temperature at time t = 0. Since y(0) = r+A·1, A = y(0) - r.

C. ***The logistic equation for population growth:*** **y' = ky(r-y)**, with <u>solutions</u> $y = r/(1+Ae^{-rkt})$, **y = 0** and **y = r**. Here r and 0 are both *equilibrium values* of y. The value y = r is a *stable equilibrium*, as a small change in y to $y(t_0) = r+\partial_{small}$ has a solution that returns toward $y(t) \approx r$. The value y = 0 is *unstable*, as a small change in y from zero leads toward y ≈ r (or toward y ¨ −∞). The constants k and r have to do with the ecological and physical situations: r is the stable population, k has to do with the rate of growth of the population, and is affected by the number of predators, the severity of the conditions, the rate of reproduction. The constant A is then determined by the initial population, since y(0) = r/(1+A), so A = [r/y(0)] - 1.

D. ***Models for the weather.*** Using a large computer, examples of past behavior, and observations from weather stations, and a specific mathematical model for weather involving a system of differential equations in many variables, one can predict the weather (future temperature in Boston wind speed in the Quad).

Examples A-C involve two quantities, y and t (time). Models of weather can involve hundreds of variables (as temperature at each station).

2. What are the assumptions of a model?

The use of any mathematical model involves assumptions — stated or hidden — about the phenomenon being modeled. They are built in to the differential equation. These assumptions may or may not be reasonable in the situation we are modeling. ONE NEEDS TO BE CAREFUL in making real life decisions based on a mathematical model. ARE THE ASSUMPTIONS VALID? There are two issues:

a. ***Does the prediction of the model fit the observed behavior?***
b. ***Do the assumptions of the model make sense?***

The first is a question of *curve fitting:*

Example: Does the solution curve for the logistic equation match the population growth observed in the real life situation?

For a mathematical model, one wants more than good curve fitting. The second question asks if the differential equation *matches well* the physical, ecological, or psychological processes assumed to be active.

Often one uses several models. A simpler model gives a rough guess as to the behavior. More complicated models adjust for observed behavior. For example, as a simple model of drug absorption by the body we use **y’ = a-by**, a variant of Newton’s Law of Cooling. Pharmacy majors taking a Kinetics course will use more complicated models.

The validity of a model needs to be tested by comparison of the solutions with actual observed behavior. This may or may not have been done by the proponents of a given mathematical model.

Examples:

A. **Exponential growth** **y’ = ky**, k > 0.

Assumption: the rate of change depends only on y, and is a constant multiple of it.

Application: Growth of amount at fixed rate of interest in bank.
Growth of population of cells.

Problem: As a population of cells grows, there are impediments to growth (lack of food, space), that make the assumption unreasonable. The assumption does not allow for variation of the environment over time.

B. **Newton’s Law of Cooling** **y’ = k(r-y),** k > 0.

Assumption: The rate of change of y is a constant multiple of the difference between the limit temperature r and y.

Application: A small object is immersed in a large medium.
A ball of molten lava falls into the ocean.

Problems: The temperature of the small object may affect the temperature of the medium. Air or water currents may change. Formation of crust on a lava requires a change in k or in the model.

C. **Logistic Equation for Population Growth**. **y' = ky(r-y)**

Assumptions: The rate of change of y depends only on the value of y. The rate of change is a constant multiple of the product of y and (r-y).

Rationale: The factor y has to do with the reproduction of the population. The factor (r-y) has to do with restraint on growth coming from predators, limitation on food supply.

Problems: The rate of change y' is assumed to adjust instantaneously to a change in y. In actual populations, there is often a delay. Seasonal change in food supply or the environment may affect population growth, but is not included in the equation. The interaction between the population y and the predators (r-y) may be more complex than assumed. Observed populations often fluctuate. The solution for this model doesn't have fluctuations.

D. **Weather models**:

Weather can be highly **unstable** in the sense that if the initial conditions are slightly changed, then the result after a day's evolution could be very different from what they would have been. This unstability has led to striking errors in computer weather prediction.

Exercises: **1**. The amount of potassium chloride y(t), t = time, y = # grams during a certain reaction, has growth described by the differential equation

$$y' = y(y-4)^2.$$

a. Why is y = 4 an equilibrium? Is it stable?
b. Draw a direction field for the equation.
c. What happens to y(t) as time increases if y(0) = 2 .
d. What are the assumptions of this model?
e. *Notice that this is not the logistic equation. How is the behavior of this population different than one modeled by the logistic equation?*

2. A medicine is given intravenously to a patient at a constant rate of 6 mg/hour, and is eliminated at a rate equal to 3y mg/hour. Here y = y(t) is a function of time.

a. Give the differential equation for y' as a function of y.
b. Draw a direction field for the equation.
c. What is the equilibrium value for y?
d. How does y(t) behave for large t in the following cases
i. y(0) = 0 mg. ii. y(0) = 1 mg. iii. y(0) = 5 mg.
e. Solve the ODE (using the solutions given above, or in your text).
f. What are the assumptions of this model?

3. A population y(t), y = # deer in thousands, t = years, satisfies y' = ky(4-y). Find the solution y(t) if y(0) = 2, and y(1) = 3. Also, formulate and answer questions similar to #2a-d,f.