

Worksheet 1.5. Group action, subgroups of symmetry groups.

Recall that G acts on a set S if there is a map $G \times S \rightarrow S: (g, s) \rightarrow g \circ s \in S$ satisfying $(gg') \circ (s) = g \circ (g' \circ s)$. The action is *transitive* if the orbit of some s is all S .

I. Consider the symmetry group $G = D_6$ of the regular hexagon X , with vertices A, B, C, D, E, F , labeled counterclockwise.

Let $r = (ABDCEF)$, the rotation $\pi/3$ radians counterclockwise. Denote by $f = (BF)(CE)$ the flip about the axis AD , and $\lambda = (AB)(CF)(DE)$ the flip about the axis connecting the midpoints of the sides DE and AB .

a. Consider the set $S = (\underline{ACE}, \underline{BDF})$, where \underline{ACE} denotes the triangle with vertices ACE . Show that G acts on S . Let $H = \text{Stab}_G(\underline{ACE})$ in this action: H is the subgroup of D_6 mapping the triangle to itself.

- i. Show that $H \cong D_3$ the full group of symmetries of the triangle (\underline{ACE})
- ii. Show that the action of G on S is transitive, by showing that r takes (\underline{ACE}) to (\underline{BDF}) . Conclude that $|D_6| = 12$.
- iii. Show that rHr^{-1} is the stabilizer of (\underline{BDF}) , using that r^{-1} takes (\underline{BDF}) to (\underline{ACE}) .
- iv. Show that H satisfies $rHr^{-1} = H$.
(as sets, this does **not** mean $rhr^{-1} = h$ for all $h \in H$.)
- v. Show that H satisfies $gHg^{-1} = H$ for all elements g of $G = D_6$. When this is true H is a *normal subgroup* of G .

b. Consider the set RECT comprised of the 3 rectangles $ABDE$, $BCEF$, and $CDFA$. Show that G acts on the set RECT .

- i. Determine the stabilizer of each rectangle. Show that the action of G on RECT is transitive.
- ii. Show that $\text{Stab}_G(ABDE)$ is isomorphic to W , the symmetries of a rectangle, and is not normal in G .

II. Consider the group K of symmetries of a tetrahedron with vertices $ABCD$.

- i. Show that the stabilizer of a vertex A is isomorphic to the group Z_3 . Identify its generator as a cycle.
- ii. Show that the orbit of A is all the vertices. Conclude that $|K| = 12$.
- iii. Show that $\text{Stab}_K A$ is not a normal subgroup of K .
- iii. Show that K has no cyclic subgroup of order 6.
- iv. Conclude that K is not isomorphic to D_6 even though their orders are the same.
- v. Can you find any other subgroups of K ? Is there any subgroup of order 6? Note that the symmetries of the triangle ABC do not form a subgroup of K .

III.** Analogous problems for the 4 dimensional analogue of the tetrahedron.