Math U575 Spring 2008 WS #$4a$ Prof. A. Iarrobino

Group actions, Burnside's theorem on # orbits. (text, Chap 29)

Let the group G act on a set S. Recall that this means there is a map $\theta : G \times S \to S, \theta(g, s) = g \circ s$ satisfying

$\theta(gg', s) = \theta(g, \theta(g', s)) : \quad so \quad (gg') \circ s = g \circ (g' \circ s).$

The action of G is *transitive* if for each $s \in S, G \circ s = S$. For any action of a group on a set S, the set is partitioned into subsets $S_1, \ldots, S_w$, called orbits, with the action of G on each set S_i being transitive.

Let w be the number of orbits of G acting on S. Denote by S^g = fix (g)=$\{s \in S \mid gs = s\}$.

Theorem (Burnside): We have

$$w = \frac{\left(\sum_{g\in G} \#S^g\right)}{|G|}$$

A concise proof is on p. 490-491 of the text. It depends on the previous result that for each orbit $S_i = G \circ s_i$ of G acting on S, we have that $Stab_G(g \circ s_i) = g \cdot Stab_G(s_i) \cdot g^{-1}$ for any $g \in G$, so their orders are equal: $|Stab_G(g \circ s_i) = |Stab_G(s_i)|$. Summing over a G-orbit S_i of s_i, we have

$$\sum_{s\in S_i} |Stab_G(s)| = |G|.$$

Now summing over the w orbits, (e.g. over all of S) we have

$$\begin{aligned} w \cdot |G| &= \sum_{s\in S} |Stab_G(s)| \\ &= \sum_{g\in G} \#S^g, \end{aligned}$$

implying (1). The last equality is from counting pairs (g, s) with $g \circ s = s$ by the elements $g \in G$ instead of by $s \in S$.

1A. Coloring a planar triangle. Let the triangle X with verices A, B, C and edges $E : E_1 = AB, E_2 = AC, E_3 = BC$ has its edges colored, using a set K of five colors. Two colorings are the same if one can move the first colored triangle in the plane such that it matches the second. For example the coloring (purple, blue, green) is the same as (green,purple, blue), but is different from (blue,purple, green) (one is not allowed to move the triangle in 3-space). How many different triangles are there?

Solution Let S be the set $K \times K \times K$ of 125 ways to color the three labeled sides. Thus (p,b,gr) $\in S$ means color the edge $E_1 = AB$ with p, E_2 with b, and E_3 with gr (colors in K). Let the group $G = Z_3 =< (ABC) >$ act on S, as it does on E: so (ABC) acts on E as (E_1, E_3, E_2).

Then Fix $((ABC)$ are colorings of the form (k,k,k) (only one color), so # (Fix((ABC))) = 5. Likewise for $(ACB) \in R$. But Fix(e)= S, so # Fix(e)= 5^3. Thus we have

$w = (125 + 5 + 5)/3 = 45$ distinct colored triangles.

Q1. Find the corresponding formula $w = (n^3 + 2n)/3$ when n colors are used on X. What is the set S?

Q2. What changes when the edges of X are colored using a set of n colors, but now two colored triangles are considered equivalent if one can be moved in 3-space so that it matches the second. So we now regard $(p, b, gr) = (b, p, gr)$. Use the group S_3, and the same set S with n^3 elements as in (1a).

Q3. Asymptotics. When the number of colors n grows large then $\lim_{n\to\infty} w = \#S/|G|$, here $n^3/3$ for planar motions of X or $n^3/6$ for spatial motions. How can you explain this? Does it make sense? Is it true for all coloring problems involving a rotation group G?

Q4. Rotations vs. symmetries. For planar life, the rotations of a triangle X are Z_3, and the symmetries (distance preserving maps X to X) are S_3, so there are more symmetries. Are there analogues for 3-D life (solids Y so that the spatial rotations of Y are not all the symmetries of Y)?