<u>Assignment 2 for MTH G131: Fall 2006</u>

<u>Due date:</u> Wednesday September 20.

Reading: Chapter 1, Sections 2.1–2.2 in Logan; also review one-variable calculus from any undergraduate text.

1). Anti-derivatives

Compute indefinite integrals of these functions:
(i) $x^{5/2}$ (ii) $\sqrt{x+2}$ (iii) $x\sqrt{x^2+4}$ (iv) $\cos(3x)$ (v) $\frac{\cos(x)}{1+\sin(x)}$

2). ODE's – analytical solutions

a) Logan, p.18, #3
b) Logan, p.18, #4
c) Logan, p.57, #1
d) Logan, p.58, #4
e) Logan, p.67, #3

3). Population models I

The logistic equation can be used to model the growth in population of the United States. By matching the known populations in the years 1790, 1850 and 1910, this leads to the following equation:

$$N(t) = \frac{210}{1 + 51.5\, e^{-0.03t}}$$

Here $N(t)$ is the population in millions and t is the number of years since 1790. [Source: "Modelling with Ordinary Differential Equations", T. P. Dreyer, CRC Press (1993)]
a) By comparing with the logistic equation $N' = (b - sN)N$; $N(0) = N_0$, find the values of the parameters b, s, N_0 in this case.
b) Plot a graph of the predicted population $N(t)$ for the period 1920 – 1980.
c) Include on your graph points to represent the following data of actual populations:
1920(106m), 1930(123m), 1940(312m), 1950(151m),
1960(179m), 1970(205m), 1980(226m).

d) For later years the model does'nt match the data too well. Why is this? Can you suggest any external factors which might have changed the parameter values during that time?

4). Population models II

It is interesting to include the effect of harvesting in the logistic model for population growth. This is done by introducing another term in the equation as follows:

$$N'(t) = (b - sN)N - h, \quad N(0) = N_0$$

Here $h \geq 0$ is a parameter which describes the (constant) rate of harvesting. (Note that this equation applies only for $N > 0$: if $N = 0$ then we assume $N' = 0$). To reduce unnecessary details, assume throughout this problem that $b = s = 1$.

a) Find the equilibrium solutions and sketch the phase-line plot. Use this to determine the long-time behavior of the solution $N(t)$. Your answer will depend on the size of h and the initial condition N_0, so consider separately the different cases. Does the solution always converge as $t \to \infty$?

b) This ODE is separable, and it can be solved exactly. It requires evaluating the following integral

$$\int \frac{1}{x - x^2 - h}\, dx$$

The solution will be different in the different cases found in (a) above. Pick one of the cases and find the solution explicitly as a function of h and N_0 [Hint: put it in a standard form and use integration tables].

5). Population models III

Here is a modified harvesting equation:

$$N'(t) = (1 - N)N - he^{-N}, \quad N(0) = N_0$$

Note that the harvesting rate now depends on N, and decreases as N increases. (Again $b = s = 1$, and $h \geq 0$ is constant.)

a) To find the equilibrium solutions, you must find the zeroes of the function $f(u) = (1 - u)u - he^{-u}$. Rather than looking for exact values, argue as follows. Sketch the two graphs $y = (1 - u)u$ and $y = he^{-u}$ and look where they cross. Show that (i) when h is small there are two equilibrium points, and identify them as attractors or repellers; (ii) when h is large there are no equilibrium points. In case (ii), what happens to $N(t)$ as $t \to \infty$?

b) Continuing the analysis in part (a), show that there is one value $h = h_c$ for which the equation has exactly one equilibrium solution. Compute the value h_c and find the corresponding equilibrium solution [Hint: at this 'critical value' the two graphs $y = (1 - u)u$ and $y = he^{-u}$ touch at one point, so the slopes of their tangent lines are also equal at this point].

c) Is the equilibrium solution in part (b) an attractor, a repeller, or neither?