<u>Assignment 4 for MTH G131: Fall 2006</u>

<u>Due date:</u> Wednesday October 4.

Reading: Chapter 1 in Logan; also review graphing functions from any undergraduate calculus text, including finding maxima, minima and points of inflection.

1). Interval of existence

Find the interval of existence for the solution $u(t)$ of the following initial value problems:

a)

$$\frac{du}{dt} = -\frac{t}{1+u}, \quad u(0) = 2$$

b)

$$\frac{du}{dt} = -\frac{t}{u^3 - u}, \quad u(0) = 2$$

2). Population model

Consider again the 'dimensionless' form of the budworm-predation model [Logan, p.38, #5] that we derived in class:

$$\frac{du}{d\tau} = \alpha u(1 - \beta u) - \frac{u^2}{1+u^2}$$

where $\alpha > 0$ and $\beta > 0$ are positive parameters (I am using 'u' in place of the variable 'n' from the text). Let $f_1(u) = \alpha(1 - \beta u)$ and $f_2(u) = \frac{u}{1+u^2}$.

a) Suppose first that α is a fixed *large* number. Sketch the graphs of f_1 and f_2 and explain why there is exactly one intersection point for all $\beta > 0$. Determine the stability of the corresponding equilibrium point u^*. Sketch the diagram showing how u^* depends on β (display β on the horizontal axis and u^* on the vertical axis).

b) The function $f_2(u)$ has a unique point of inflection P for positive u. Find the coordinates of P, and find the equation of the tangent line at P.

c) There is a positive value α_c with the following property: when $\alpha > \alpha_c$, the graphs of f_1 and f_2 have exactly one intersection point for all β; when

$\alpha < \alpha_c$, the graphs of f_1 and f_2 have three intersection points for some interval of values of β. By relating the graph of f_1 to the tangent line through P, determine this value α_c.

3). Bifurcation diagrams

In each case find the values of h at which bifurcations occur, and classify them as saddle-node, transcritical, supercritical pitchfork or subcritical pitchfork. Sketch the bifurcation diagram of u^* against h.

a) $\frac{du}{dt} = h - 3u^2$
b) $\frac{du}{dt} = hu - \frac{u}{1+u}$
c) $\frac{du}{dt} = hu - \sinh(u)$

4). Epidemic model

A fixed population at time t has $x(t)$ healthy people, $y(t)$ sick people and $z(t)$ dead people. Healthy people get sick at a rate proportional to the product of x and y, and sick people die at a fixed rate. This leads to the set of equations

$$\frac{dx}{dt} = -kxy, \quad \frac{dy}{dt} = kxy - my, \quad \frac{dz}{dt} = my$$

a) Using the fact that $x + y + z = N$ is constant, derive the equation $m\dot{x} = -kx\dot{z}$.

b) Derive a first order ODE for $z(t)$, involving k, m, N and $x(0)$, the initial value of x. By rescaling z and t, show that this reduces to the dimensionless form

$$\frac{du}{d\tau} = \alpha - \beta u - e^{-u}$$

where $\alpha \geq 1$ and $\beta > 0$.

c) Deduce that x, y, z converge to constant values as $t \to \infty$. If $x(0) < \frac{m}{k}$ show that $y(t)$ is always decreasing. What happens if $x(0) > \frac{m}{k}$?