Assignment 5 for MTH G131: Fall 2006

Due date: Wednesday October 11.

Reading: Logan, Chapter 1; undergraduate linear algebra text.

1). [Modeling: overdamped pendulum]

A rigid pendulum of length L can rotate in a plane about one fixed end. The other end carries a mass M. There is a fixed external torque Γ which tries to make the pendulum rotate about its fixed end. The equation of motion of the pendulum is

$$ML^2\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + MLg\sin(\theta) = \Gamma$$

where θ is the angle between the pendulum and the downward vertical, b is a damping constant and g is the acceleration due to gravity. Considering the overdamped case where b is very large leads to the simplified equation of motion

$$b\frac{d\theta}{dt} + MLg\sin(\theta) = \Gamma$$

a) By rescaling θ and t rewrite this in the dimensionless form

$$\theta' = \gamma - \sin(\theta)$$

b) Find the equilibrium solutions assuming that $|\gamma| < 1$, and determine $\lim_{t\to\infty}\theta(t)$. Indicate the positions of the equilibrium solutions on a circle, where $\theta = 0$ is at the bottom.

c) Determine the long-time behavior of $\theta(t)$ in the case $\gamma > 1$.

2). [Matrices]

a) Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

For each eigenvalue, find a non-zero eigenvector.

b) Find eigenvalues and eigenvectors for the matrix $B = 2A - 3I$ where I is the 2×2 identity matrix. [Hint: you can figure this out from the answer to part (a), no need to calculate determinants].

3). [Matrices]
Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

a) Calculate A^2 and A^3.

b) Verify by induction the formula

$$A^n = \begin{pmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{pmatrix}$$

[That is, assume it holds for n, then calculate $A^{n+1} = AA^n$ and verify the formula for $n + 1$.]

c) The exponential of A is defined by the series

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Use the formula from (b) to calculate e^A.

4). [Matrices]

a) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

b) Use your result from part (a) to find the inverse of the matrix

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5). [Matrices]
Consider two distinct numbers, a and b. Define the function

$$f(t) = \text{Det} \begin{bmatrix} 1 & 1 & 1 \\ t & a & b \\ t^2 & a^2 & b^2 \end{bmatrix}$$

a) Show that $f(t)$ is a quadratic function. What is the coefficient of t^2?

b) Explain why $f(a) = f(b) = 0$. Conclude that $f(t) = k(t-a)(t-b)$ for some constant k. Use the results from (a) to find k.

c) For which values of t is the matrix invertible?