

Practice Problems 1 for MTH G131: Fall 2006

1). Find the general solution of the following ODE:

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$$

2). Find the solution of the following initial value problem, and determine the interval of existence:

$$\frac{dx}{dt} = \frac{2t}{1+2x}, \quad x(2) = 0$$

3). Newton's law of cooling states that the surface temperature of an object changes at a rate proportional to the difference between the temperature of the object and that of its surroundings. Let $u(t)$ be the object's temperature at time t , and let u_0 be the fixed temperature of the surroundings. Then

$$\frac{du}{dt} = k(u - u_0)$$

where k is constant.

a) Find the solution satisfying the initial condition $u(0) = u_0$.

b) Suppose the temperature of a cup of tea is 200 F when freshly poured. One minute later it has cooled to 190 F in a room at 70 F. How long must elapse before the coffee reaches a temperature of 150 F ?

4). Find eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

5). Find the general solution of the ODE

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

6). Find the solution of the following system of linear ODE's:

$$\frac{dx}{dt} = 4x - 3y$$

$$\frac{dy}{dt} = 6x - 7y$$

with initial conditions $x(0) = 1$, $y(0) = -1$.

7). Determine if the critical point $(0, 0)$ is stable, asymptotically stable, or unstable.

a)
$$\frac{dx}{dt} = 2x - y, \quad \frac{dy}{dt} = 3x - 2y$$

b)
$$\frac{dx}{dt} = x - 5y, \quad \frac{dy}{dt} = x - 3y$$

8). Each of the following systems has a critical point at the origin. In each case show that the system is almost linear (in the sense defined in class), and if possible determine whether the critical point is asymptotically stable or unstable.

a)
$$\frac{dx}{dt} = x + x^2 + y^2, \quad \frac{dy}{dt} = y - xy$$

b)
$$\frac{dx}{dt} = (1 + x) \sin y, \quad \frac{dy}{dt} = 1 - x - \cos y$$

9). The equation of motion of a spring-mass system with damping is

$$m \frac{d^2u}{dt^2} + b \frac{du}{dt} + ku = 0$$

where m, b, k are positive. Divide across by m , let $\alpha = k/m$ and $\beta = b/m$, and write this as a system of two first order equations for $x = u$ and $y = du/dt$. Show that $x = 0, y = 0$ is a critical point, and analyze the stability of the critical point as a function of the parameters α and β .

10). In each case find the values of h at which bifurcations occur, and classify them as saddle-node, transcritical, supercritical pitchfork or subcritical pitchfork. Sketch the bifurcation diagram of u^* against h .

a)

$$\frac{du}{dt} = h + u - \ln(1 + u)$$

b)

$$\frac{du}{dt} = hu - \ln(1 + u)$$

c)

$$\frac{du}{dt} = hu - 4u^3$$

11). A simple model for the action of a biochemical switch envisions a gene which can be ‘switched on’ when the concentration of a biochemical agent exceeds a certain threshold. If $x(t)$ is the concentration of the gene product and h is the concentration of the biochemical agent then the model is (after rescaling)

$$\frac{dx}{dt} = h - ax + \frac{x^2}{1 + x^2}$$

where $a > 0$ (the last term is an autocatalytic or positive feedback term).

a) Show that if $h = 0$ there are two positive fixed points u^* if $a < a_c$, where a_c is to be determined.

b) Suppose initially $x(0) = 0$, and suppose that h is slowly increased from zero. What happens to $x(t)$?

c) What happens if h then slowly goes back to zero? Does $x(t)$ return to zero? i.e. does the gene ‘turn off’ again?