

## Solutions for Practice Problems 2 for MTH 131: Fall 2006

1). Evaluate the following limits:

$$(i) \lim_{x \rightarrow 1} \frac{\sqrt{1-x^2}}{1-x} \quad (ii) \lim_{x \rightarrow 0} \frac{x - \sin(x)}{x - \tan(x)}$$

Answers: (i)  $+\infty$  (ii)  $-1/2$ .

2). Solve the following initial value problem:

$$\frac{dy}{dx} = 2x(y+1), \quad y(0) = 0$$

Answers:  $y = e^{x^2} - 1$

3). Compute the indefinite integrals of these functions:

$$(i) x^{-5/2} \quad (ii) \frac{x}{(x-1)(2x+3)} \quad (iii) x\sqrt{x^2 + 2x + 5}$$

Answers: (i)  $-2/3 x^{-3/2} + C$  (ii)  $1/5 \ln(x-1) + 3/10 \ln(2x+3) + C$

4). Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{2xy^2}{1+x^2}, \quad y(x_0) = y_0$$

Find the interval of existence for the cases where  $y_0 > 0$  and  $y_0 < 0$ .

Answers:  $y = \left( \frac{1}{y_0} - \ln \frac{1+x^2}{1+x_0^2} \right)^{-1}$ . For  $y_0 > 0$  interval of existence is  $x^2 < (1+x_0^2)e^{1/y_0} - 1$ , inequality reversed for  $y_0 < 0$ .

5). By changing variables to  $y = u^{1-n}$ , show that the ODE

$$\frac{du}{dt} = a(t)u + g(t)u^n$$

is transformed into a linear ODE for  $y$ .

Answers:  $y' = (1-n)a(t)y + (1-n)g(t)$ .

6). Here is a modified harvesting equation:

$$N'(t) = (1 - N)N - h(1 + N^3), \quad N(0) = N_0$$

The harvesting rate depends on  $N$ , and increases as  $N$  increases ( $h \geq 0$  is constant.)

a) To find the equilibrium solutions, you must find the zeroes of the function  $f(u) = (1 - u)u - h(1 + u^3)$ . Rather than looking for exact values, argue as follows. Sketch the two graphs  $y = (1 - u)u$  and  $y = h(1 + u^3)$  and look where they cross. Show that (i) when  $h$  is small there are two equilibrium points, and identify them as attractors or repellers; (ii) when  $h$  is large there are no equilibrium points. In case (ii), what happens to  $N(t)$  as  $t \rightarrow \infty$ ?

b) Continuing the analysis in part (a), show that there is one value  $h = h_c$  for which the equation has exactly one equilibrium solution. Compute the value  $h_c$  and find the corresponding equilibrium solution.

*Answers:*  $h_c$  and  $u_c$  are related by equation  $u^2 - 2u + 3h = 0$ , then  $u_c$  is solution of  $u^4 - 2u^3 - 2u + 1 = 0$ .

c) Is the equilibrium solution in part (b) an attractor, a repeller, or neither?

7). Compute the Taylor polynomials of  $x^{3/2}$  to the third order at the point  $x = 1$ .

*Answers:*  $1 + 3/2(x - 1) + 3/8(x - 1)^2 - 1/16(x - 1)^3$ .

8). In each case find the values of  $h$  at which bifurcations occur, and classify them as saddle-node, transcritical, supercritical pitchfork or subcritical pitchfork. Sketch the bifurcation diagram of  $u^*$  against  $h$ .

a)

$$\frac{du}{dt} = h + 3u^2$$

*Answers:*  $h = 0$  saddlenode.

b)

$$\frac{du}{dt} = hu^2 - \frac{u}{1 + u}$$

*Answers:*  $h = -4$ , pitchfork.

9). Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$$

For each eigenvalue, find a non-zero eigenvector.

*Answers:*  $\lambda_1 = 3 + \sqrt{12}$ ,  $v_1 = \begin{pmatrix} 2 \\ 2 + \sqrt{12} \end{pmatrix}$ .  $\lambda_2 = 3 - \sqrt{12}$ ,  $v_2 = \begin{pmatrix} 2 \\ 2 - \sqrt{12} \end{pmatrix}$ .

10). Plot the vector function where  $-\infty < t < \infty$ :

$$\mathbf{r}(t) = \begin{pmatrix} t^2 \\ 1 - t \end{pmatrix} e^{2t}$$

11). Find the general solution and sketch the phase portrait. Indicate linear orbits (if any) and indicate direction of solution curves:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix}$$

*Answers:*  $ae^{-11t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + be^{-4t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ .

12). Find a two-dimensional system whose matrix has eigenvalues  $\lambda = 2$  and  $\lambda = -4$ .

*Answers:*  $x' = 2x$ ,  $y' = -4y$ .

13). Each of the following linear systems has a critical point at the origin. In each case determine whether the critical point is stable, asymptotically stable, or unstable.

a)  $\frac{dx}{dt} = 2x - 5y$ ,  $\frac{dy}{dt} = x + 2y$

b)  $\frac{dx}{dt} = 3x - y$ ,  $\frac{dy}{dt} = 4x - y$

*Answers:* both unstable.

**14).** Each of the following systems has a critical point at the origin. In each case show that the system is almost linear (in the sense defined in class), and if possible determine whether the critical point is asymptotically stable or unstable.

$$\mathbf{a)} \quad \frac{dx}{dt} = x - 2y + \sqrt{x^2 + y^2}, \quad \frac{dy}{dt} = 3x + 2y - xy$$

$$\mathbf{b)} \quad \frac{dx}{dt} = x + 2y + xy, \quad \frac{dy}{dt} = -2x + y - x^3y$$

*Answers:* (a) ERROR: nonlinear term is not 'small'! (b) unstable.

**15).** Determine the equilibria and stability of the system

$$\frac{dx}{dt} = y + (1 - x)(2 - x), \quad \frac{dy}{dt} = y - kx^2$$

as a function of the parameter  $k > 0$ . Sketch the phase diagram.

*Answers:* for  $k > 1/8$  no critical points. For  $k < 1/8$  two unstable critical points at  $x = \frac{3 \pm \sqrt{1 - 8k}}{2(1+k)}$ .

**16).** Find the general solution of the equations

**a)**  $u'' + 9u = 0$

**b)**  $u'' - 2u' = 0$

**c)**  $u'' + u' = 9e^{-t}$

*Answers:* (a)  $a \cos 3t + b \sin 3t$ , (b)  $ae^{2t} + b$ , (c)  $(a - 9t)e^{-t} + b$ .

**17).** The following integrals may or may not exist, due to singularities of the functions being integrated, and/or because the domain of integration is infinite. In each case, decide if the integral exists (you do NOT need to evaluate the integrals!). If you think the integral exists, you should find a positive integrable function that upper bounds the given function. If you think it doesn't exist, you should find a positive non-integrable function that lower bounds the given function. [Hint: look at how the function behaves when  $x$  is near the singularity, or when  $x$  is very large.]

$$\int_0^{\pi/2} \frac{\sin x}{x^2} dx, \quad \int_1^{\infty} (1 + x^4)^{-1} dx$$

*Answers:* (a)  $+\infty$ , dominates  $\frac{2}{\pi x}$  (b) finite, dominated by  $x^{-4}$ .

**18).** Solve the following initial value problems using Laplace transforms:

**a)**  $u'' - u' - 6u = 0, \quad u(0) = 2, \quad u'(0) = -1$

**b)**  $u'' + 0.4u' + 2u = 1 - h_5(t), \quad u(0) = 0, \quad u'(0) = 0$

**c)**  $u'' + 4u = \sin t + h_\pi(t) \sin(t - \pi), \quad u(0) = 0, \quad u'(0) = 0$

**19).** Invert the Laplace transforms

**a)** 
$$\frac{2s + 1}{4s^2 + 4s + 5}$$

**b)** 
$$\frac{e^{2-4s}}{2s - 1}$$