

**MTH U481 : SPRING 2009: PRACTICE PROBLEMS FOR MIDTERM**

1). Find the mean and variance of the random variable  $Y$  with the following pdf:

$$f_Y(y) = \begin{cases} y/2 & 0 \leq y \leq 1 \\ 1/2 & 1 \leq y \leq 5/2 \\ 0 & \text{elsewhere} \end{cases}$$

*Answer:*  $71/48$ ;  $0.3746$

2). A continuous random variable  $X$  has the following cdf:

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & \text{if } x \leq -1 \\ (1+x)^2/4 & \text{if } -1 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

a). Sketch the graph of  $F_X$ , and calculate  $P(X \leq 1/2)$ .

*Answer:*  $F_X(1/2) = 9/16$

b). Find  $P(|X| \geq 1/2)$ .

*Answer:*  $F_X(-1/2) + 1 - F_X(1/2)$

c). Calculate the pdf  $f_X$ .

*Answer:*  $(1+x)/2$  for  $-1 \leq x \leq 1$

3).  $X$  and  $Y$  are discrete random variables, each with two possible values, namely 1 and  $-1$ . So the joint pdf of the pair  $X, Y$  is a  $2 \times 2$  table. Suppose that  $P(X = 1, Y = 1) = 1/3$ , and also  $E[X] = E[Y] = 0$ .

a). Find the other three entries in the joint pdf.

*Answer:*  $P(-1, -1) = 1/3, P(-1, 1) = P(1, -1) = 1/6$

b). Calculate  $VAR[X]$  and  $VAR[Y]$ .

*Answer:*  $VAR[X] = VAR[Y] = 1$

c). Calculate  $COV[X, Y]$  and the correlation coefficient  $\rho(X, Y)$ .

*Answer:*  $\rho(X, Y) = 1/3$

d). Are  $X$  and  $Y$  independent? Explain.

*Answer:* No, since  $\rho(X, Y) \neq 0$

4). Two independent components in a circuit have lifetimes  $T_1$  and  $T_2$ , each with the same pdf:

$$f_{T_1}(t) = f_{T_2}(t) = \begin{cases} 3e^{-3t} & t \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

a). Find the expected lifetime of each component.

*Answer:*  $1/3$

The circuit is designed to work until both components fail. Let  $S$  be the lifetime of the circuit.

b). Explain why  $P(S \leq s) = P(T_1 \leq s \text{ and } T_2 \leq s)$ .

*Answer:* the events  $\{S \leq s\}$  and  $\{T_1 \leq s\} \cap \{T_2 \leq s\}$  are the same (to show this, point out that the first event implies the second, and the second implies the first, hence they are equal).

c). Use (b) to find the cdf of  $S$ , and hence find the expected value of  $S$ .

*Answer:*  $F_S(s) = (1 - e^{-3s})^2$ ;  $E[S] = 1/2$

5). A point  $(X, Y)$  is chosen randomly and uniformly from the parallelogram with corners at the four points  $(0, 0)$ ,  $(2, 1)$ ,  $(1, 3)$ ,  $(3, 4)$ . Find the probability that  $X$  is greater than  $Y$ , and calculate the marginal pdf's of  $X$  and  $Y$ . [Hint: the area of the parallelogram spanned by the vectors  $(u_1, u_2)$  and  $(v_1, v_2)$  is  $|u_1v_2 - u_2v_1|$ ].

*Answer:*  $P(X > Y) = 1/4$ .

$$f_X(x) = \begin{cases} x/2 & 0 \leq x \leq 1 \\ 1/2 & 1 \leq x \leq 2 \\ (3-x)/2 & 2 \leq x \leq 3 \end{cases}, \quad f_Y(y) = \begin{cases} y/3 & 0 \leq y \leq 1 \\ 1/3 & 1 \leq y \leq 3 \\ (4-y)/3 & 3 \leq y \leq 4 \end{cases}$$

6). Consider two independent random numbers  $U$  and  $V$ . Each of them is uniformly distributed over the interval  $[0, 1]$ .

a). Find the probability that  $U^2 \leq 1/2$ .

*Answer:*  $1/\sqrt{2}$

b). Find the expected value and variance of  $Y = U - 3V$ .

*Answer:*  $-1$ ;  $5/6$

c). Find the probability that  $U + 2V \leq 2$ .

*Answer:*  $3/4$

7). The pdf for  $X$  equals  $c$  for  $0 \leq x \leq 2$ , and equals  $ke^{-x}$  for  $x \geq 2$ . Assuming the pdf is continuous find  $k$  and  $c$ , and calculate  $E[X]$  and  $VAR[X]$ .

*Answer:*  $c = 1/3$ ,  $k = e^2/3$ ,  $E[X] = 5/3$ ,  $VAR[X] = 13/9$

8). A fair coin is tossed twice, the number of Heads is  $X$ . A biased coin is then tossed  $X$  times, and the number of Heads for the biased coin is  $Y$ . If  $p$  is the probability of Heads for the biased coin, write down the joint pdf for  $X$  and  $Y$ . Compute the marginals for  $X$  and  $Y$ . If  $p = 1/3$  find the correlation coefficient of  $X$  and  $Y$ .

*Answer:*

$$P(0,0) = 1/4, \quad P(1,0) = (1-p)/2, \quad P(2,0) = (1-p)^2/4,$$

$$P(1,1) = p/2, \quad P(2,1) = p(1-p)/2, \quad P(2,2) = p^2/4$$

all others are zero. The marginals are

$$P(X=0) = 1/4, \quad P(X=1) = 1/2, \quad P(X=2) = 1/4$$

$$P(Y=0) = (2-p)^2/4, \quad P(Y=1) = p(2-p)/2, \quad P(Y=2) = p^2/4$$

Correlation coefficient is  $\sqrt{p/(2-p)}$ .

9). A rare disease occurs only in 1 person out of every 10,000. If a population of 1000 people is tested, what is the probability that one or more people have the disease?

**Hint:** use the Poisson distribution to estimate the answer.

*Answer:* 0.095

10). A manufacturing process produces widgets whose length is a random variable, with uniform distribution between 2.0 and 4.0. Let  $\bar{X}$  be the average length of a batch of 400 widgets.

a). Find the mean and variance of  $\bar{X}$ .

*Answer:* 3;  $1/20\sqrt{3}$

b). Use the Central Limit Theorem to estimate the probability that  $\bar{X}$  exceeds 3.08.

*Answer:*  $1 - F(2.77)$

11). A certain brand of car has a defect which may cause it to break down after 50,000 miles. It is known that this problem occurs for one out of every five of these cars. A rental company owns 400 of these cars. Use the normal approximation to estimate the probability that more than 90 of their cars will break down because of this defect.

*Answer:*  $1 - F(1.25)$