Assignment # 11: $\boxed{\text{SOLUTIONS}}$

$\boxed{1}$ (p. 355)

① Likelihood function

$$L(\theta) = P(X_1=1)P(X_2=0)P(X_3=1)P(X_4=1)P(X_5=0)P(X_6=1)P(X_7=1)P(X_8=0)$$
$$= \theta^5 (1-\theta)^3.$$

$$\ln L(\theta) = 5\ln\theta + 3\ln(1-\theta)$$

$$\frac{d}{d\theta}\ln L(\theta) = \frac{5}{\theta} - \frac{3}{1-\theta} = 0 \Rightarrow 5-5\theta = 3\theta \Rightarrow \boxed{\hat{\theta} = \frac{5}{8}}.$$

④ Data $k_1, k_2, \ldots, k_n$.

$$L(\theta) = \frac{\theta^{2k_1} e^{-\theta^2}}{k_1!} \frac{\theta^{2k_2} e^{-\theta^2}}{k_2!} \cdots \frac{\theta^{2k_n} e^{-\theta^2}}{k_n!}$$

$$= \frac{\theta^{2(k_1+k_2+\cdots+k_n)} e^{-n\theta^2}}{k_1!\cdots k_n!}$$

$$\ln L(\theta) = 2(k_1+\cdots+k_n)\ln\theta - n\theta^2 - \ln(k_1!\cdots k_n!)$$

$$\frac{d}{d\theta}\ln L(\theta) = \frac{2(k_1+\cdots+k_n)}{\theta} - 2n\theta = 0$$

$$\Rightarrow \hat{\theta} = \sqrt{\frac{k_1+\cdots+k_n}{n}}.$$

7 $f(y,\theta) = \theta y^{\theta-1} \qquad 0 \le y \le 1$

Measure: $y_1 = 0.77$, $y_2 = 0.82$, $y_3 = 0.92$, $y_4 = 0.94$, $y_5 = 0.98$.

Likelihood funct:

$$L(\theta) = \theta y_1^{\theta-1} \, \theta y_2^{\theta-1} \cdots \theta y_5^{\theta-1}$$

$$= \theta^5 (y_1 \cdots y_5)^{\theta-1}$$

$$\ln L(\theta) = 5 \ln \theta + (\theta-1) \ln(y_1 \cdots y_5)$$

$$\frac{d}{d\theta} \ln L(\theta) = \frac{5}{\theta} + \ln(y_1 \cdots y_5) = 0$$

$$\Rightarrow \hat{\theta} = \frac{-5}{\ln(y_1) + \cdots + \ln(y_5)}$$

Data $\Rightarrow$ $\hat{\theta} = \dfrac{-5}{\sum_i \ln y_i} = 7.996.$

3 $X \sim \text{Poisson}(\lambda) \qquad Y \sim \text{Poisson}(2\lambda).$

$\Rightarrow E[X] = VAR[X] = \lambda \qquad E[Y] = VAR[Y] = 2\lambda.$

$$\hat{\lambda} = aX + \frac{1-a}{2} Y.$$

$\Rightarrow E[\hat{\lambda}] = a\lambda + \frac{1-a}{2} 2\lambda = \lambda. \Rightarrow$ unbiased.

(4)

$$VAR[\hat{\lambda}] = VAR[aX] + VAR\left[\frac{1-a}{2} Y\right] \quad \text{(independent)}$$

$$= a^2 VAR[X] + \left(\frac{1-a}{2}\right)^2 VAR[Y]$$

$$= a^2 \lambda + \left(\frac{1-a}{2}\right)^2 2\lambda$$

$$= \frac{\lambda}{2}\left[2a^2 + 1 - 2a + a^2\right]$$

$$= \frac{\lambda}{2}\left[3a^2 - 2a + 1\right].$$

~~Minimize:~~ $\Rightarrow MSE[\hat{\lambda}] = \frac{\lambda}{2}(3a^2 - 2a + 1)$

Minimize:

$$\frac{d}{da} MSE[\hat{\lambda}] = \frac{\lambda}{2}(6a - 2) = 0 \Rightarrow \boxed{a = \frac{1}{3}}.$$

$\boxed{1}$ $X = \mu + \sigma Z$ μ = true weight

σ = st. dev = 0.1

Sample mean $\hat{\mu} = \frac{1}{5}(3.142 + 3.163 + 3.155 + 3.150 + 3.141) = 3.1502$

a) $\mu = \hat{\mu} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ $\alpha = 0.05$; $n = 5 \Rightarrow Z_{\alpha/2} = 1.96$

$\Rightarrow \mu = 3.1502 \pm 1.96 \frac{0.1}{\sqrt{5}}$

$= 3.15 \pm 0.088$

b) $\alpha = 0.01 \Rightarrow Z_{\alpha/2} = 2.58$

$\Rightarrow \mu = 3.15 \pm 2.58 \frac{0.1}{\sqrt{5}}$

$= 3.15 \pm 0.115$

c) $\mu \leq \hat{\mu} + Z_\alpha \frac{\sigma}{\sqrt{n}}$ $\alpha = 0.05 \Rightarrow Z_\alpha = 1.64$

$\mu \leq 3.15 + 1.64 \frac{0.1}{\sqrt{5}}$

$\mu \leq 3.224$

d) $\mu = \hat{\mu} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = \hat{\mu} \pm 1.96 \frac{0.1}{\sqrt{n}}$

$\Rightarrow$ want $(1.96) \frac{0.1}{\sqrt{n}} \leq 0.01 \Rightarrow \sqrt{n} \geq 19.6 \Rightarrow n \geq 385$.

2 $X = \#$ supporting gun control.

$X \sim Bin(n, p)$.

$\hat{p} = \frac{X}{n}$

$p = \hat{p} \pm Z_{\alpha/2} \frac{1}{2\sqrt{n}}$

"19 out of 20 cases" $\Rightarrow$ 95% C.I $\Rightarrow \alpha = 0.05 \Rightarrow Z_{\alpha/2} = 1.96$

$n = 10,000$

$\Rightarrow p = \hat{p} \pm 1.96 \frac{1}{2\sqrt{10,000}} = \hat{p} \pm 0.0098$

$\Rightarrow$ accuracy is $\pm 0.98\%$ $\Rightarrow x = 0.98$.

3 (3) $X \sim N(\mu, \sigma^2)$

$\sigma = 14.3$ $\qquad \mu = \hat{\mu} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$.

$\alpha = 0.05 \Rightarrow \mu = \hat{\mu} \pm 1.96 \frac{14.3}{\sqrt{n}}$.

Length of C.I $= 2(1.96) \frac{14.3}{\sqrt{n}} < 3.06 \Rightarrow n \geq 336$.

(8) $X = \#$ contaminated salads.

$X \sim Bin(n, p)$ $\qquad n = 220$ $\qquad p =$ unknown

$\hat{p} = \frac{X}{n} = \frac{179}{220} = 0.8136$

90% C.I $\Rightarrow \alpha = 0.1 \Rightarrow Z_{\alpha/2} = 1.64$

$\Rightarrow p = \hat{p} \pm Z_{\alpha/2} \frac{1}{2\sqrt{n}} = 0.8136 \pm 1.64 \frac{1}{2\sqrt{220}} = \cancel{0.03}$

$= 0.814 \pm 0.055$

$= (0.759, 0.869)$