MTH 481:

Assignment # 5: $\boxed{\text{SOLUTIONS}}$

(p.172)

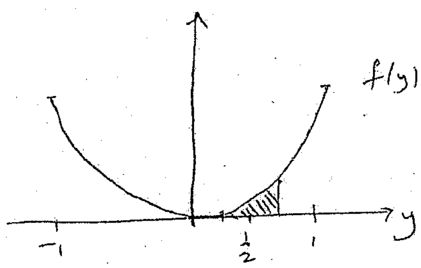
1. $P(0 \le Y \le \frac{1}{2}) = \int_0^{\frac{1}{2}} f(y)\,dy = \int_0^{\frac{1}{2}} 4y^3\,dy = y^4 \Big|_0^{\frac{1}{2}} = (\frac{1}{2})^4 = \frac{1}{16}.$

2. $P(\frac{3}{4} \le Y \le 1) = \int_{3/4}^{1} f(y)\,dy = \int_{3/4}^{1} (\frac{2}{3} + \frac{2}{3}y)\,dy = \frac{2}{3}y + \frac{1}{3}y^2 \Big|_{3/4}^{1}$

$= \frac{2}{3} + \frac{1}{3} - (\frac{2}{3}\cdot\frac{3}{4} + \frac{1}{3}(\frac{3}{4})^2)$

$= 1 - \frac{1}{2} - \frac{3}{16} = \frac{5}{16}.$

3.



$P(|Y - \frac{1}{2}| < \frac{1}{4}) = P(\frac{1}{4} < Y < \frac{3}{4})$

$= \int_{\frac{1}{4}}^{3/4} \frac{3}{2}y^2\,dy$

$= \frac{1}{2}y^3 \Big|_{1/4}^{3/4}$

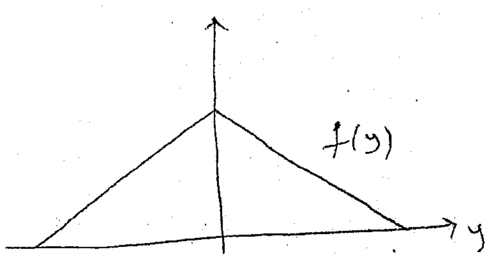
$= \frac{1}{2}\left[(\frac{3}{4})^3 - (\frac{1}{4})^3\right] = \frac{13}{64}.$

4. $P(Y > 1) = \int_1^3 \frac{1}{9}y^2\,dy = \frac{1}{27}y^3 \Big|_1^3 = \frac{1}{27}(27-1) = \frac{26}{27}.$

7. c.d.f. $F(y) = P(Y \le y) = \begin{cases} y^4 & 0 \le y \le 1 \\ 0 & y < 0 \\ 1 & y > 1 \end{cases}$

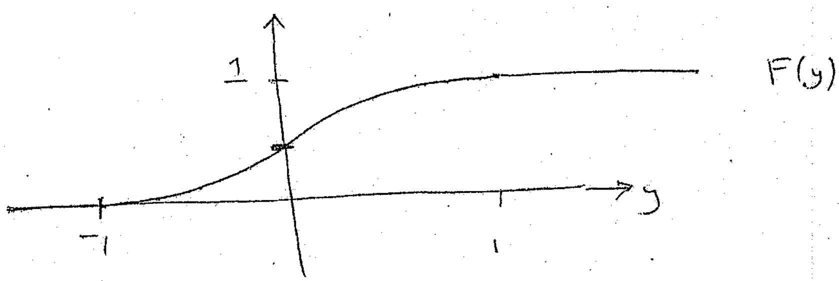
$\Rightarrow P(0 \le Y \le \frac{1}{2}) = F(\frac{1}{2}) - F(0) = (\frac{1}{2})^4 - 0 = \frac{1}{16}.$

9.



$f(y) = \begin{cases} 0 & y < -1 \\ 1+y & -1 \le y \le 0 \\ 1-y & 0 \le y \le 1 \\ 0 & y > 1 \end{cases}$

$$F(y) = \begin{cases} 0 & y < -1 \\ y + \frac{y^2}{2} + \frac{1}{2} & -1 \le y \le 0 \\ y - \frac{y^2}{2} + \frac{1}{2} & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$



(11) $F(y) = \begin{cases} 0 & y < 1 \\ \ln y & 1 \le y \le e \\ 1 & y > e. \end{cases}$ $\quad e = 2.718\ldots$

a) $P(Y<2) = F(2) = \ln 2.$

b) $P(2 < Y \le 2.5) = F(2.5) - F(2) = \ln(2.5) - \ln(2) = \ln(1.25)$

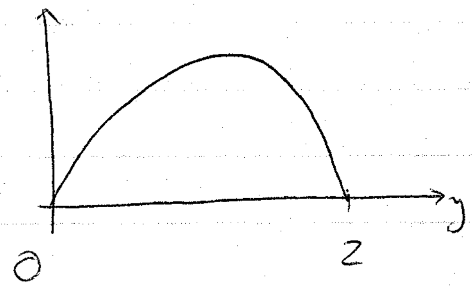
c) same since $P(Y=2.5) = 0.$

d) $f_Y(y) = \begin{cases} 0 & y<1 \\ \frac{1}{y} & 1 \le y \le e \\ 0 & y > e. \end{cases}$

3

2 $f(y) = \begin{cases} c(2y - y^2) & 0 \le y \le 2 \\ 0 & \text{else} \end{cases}$

a)



$$\text{Area} = \int_0^2 f(y)\,dy = 1 \Rightarrow c\int_0^2 (2y - y^2)\,dy = c\left(y^2 - \frac{y^3}{3}\right)\Big|_0^2 = 1.$$

$$\Rightarrow \boxed{c = \frac{3}{4}}$$

b) $P(0 \le Y \le 1) = \frac{1}{2}$ by symmetry

c) $\underline{F(y)}$: $y < 0 \Rightarrow F(y) = 0$

$y > 2 \Rightarrow F(y) = 1$

$0 < y < 2 \Rightarrow F(y) = \int_0^y f(u)\,du = \frac{3}{4}\int_0^y (2u - u^2)\,du$

$= \frac{3}{4}\left(y^2 - \frac{y^3}{3}\right).$

3

Y	0	1
prob	$\frac{5}{6}$	$\frac{1}{6}$

$P(Y = 1) = \frac{6}{36} = \frac{1}{6}$

4 (p. 185, #8).

a) $E[Y] = \int_0^1 y\, 3(1-y)^2\,dy = \frac{1}{4}$

b) $E[Y] = \int_0^\infty y\, 4y\, e^{-2y}\,dy = 4\,\frac{2!}{2^3} = 1.$

c) $E[Y] = \int_0^1 y \cdot \frac{3}{4}\,dy + \int_2^3 y \cdot \frac{1}{4}\,dy = 1.$

d) $E[Y] = \int_0^{\pi/2} y \sin y\,dy = 1.$

5 (p. 185, #23)

$X = \#$ games played $\quad \mathrm{Ran}(X) = \{\cancel{3}\, 4, 5, 6, 7\}$

$P(X=4) = 2\left(\frac{1}{2}\right)^4$ (one team wins 4 games in a row)

$P(X=5) = 2\binom{4}{3}\left(\frac{1}{2}\right)^5$ (one team wins 3 out of first 4 games)

$P(X=6) = 2\binom{5}{3}\left(\frac{1}{2}\right)^6$ (— " — 3 — " — 5 games)

$P(X=7) = 2\binom{6}{3}\left(\frac{1}{2}\right)^7$ (— " — 3 — " — 6 — " —)

$\Rightarrow E[X] = 4\,P(X=4) + 5\,P(X=5) + 6\,P(X=6) + 7\,P(Y=7)$

$= \frac{93}{16} = 5.8125$

6 (p. 192)

#30. $E[Q] = E\left[2\left(1-e^{-2Y}\right)\right]$

$= \int_0^\infty 2\left(1-e^{-2y}\right) 6e^{-6y}\,dy$

$= \frac{1}{2}$

#31. Volume $= 5Y^2$

$E[\text{Volume}] = E[5Y^2]$

$= 5\,E[Y^2] = 5\int_0^1 y^2\, 6y(1-y)\,dy = \frac{3}{2}$

$\boxed{6}$ (p. 198, #2)

$$E[Y] = \int_0^1 y \tfrac{3}{4}\,dy + \int_2^3 y \tfrac{1}{4}\,dy = 1$$

$$E[Y^2] = \int_0^1 y^2 \tfrac{3}{4}\,dy + \int_2^3 y^2 \tfrac{1}{4}\,dy$$

$$= \frac{11}{6}$$

$$\Rightarrow VAR[Y] = E[Y^2] - E[Y]^2 = \frac{5}{6}$$