MTH 581 : Sp 08

$\boxed{\text{Assignment \#2:}}$ SOLUTIONS

$\boxed{1}$ (p. 161)

(#7) Let $X =$ position after 4 steps.

$\text{Ran } X = \{-4, -2, 0, 2, 4\}$

pdf is same as for fair coin toss: $\left\{ \begin{array}{l} \text{say Right = Heads} \\ \text{Left = Tails} \end{array} \right.$

X	-4	-2	0	2	4
prob	$\left(\frac{1}{2}\right)^4$	$\binom{4}{1}\left(\frac{1}{2}\right)^4$	$\binom{4}{2}\left(\frac{1}{2}\right)^4$	$\binom{4}{3}\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^4$

all Tails

all Heads

3 Heads, 1 Tail

(#8) Now it corresponds to a biased coin tossed 4 times:

$\left. \begin{array}{l} \text{Right = Heads} \\ \text{Left = Tails} \end{array} \right\}$ $P(H) = \frac{2}{3}$, $P(T) = \frac{1}{3}$.

X	-4	-2	0	2	4
prob	$\left(\frac{1}{3}\right)^4$	$\binom{4}{1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)$	$\binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$	$\binom{4}{3}\frac{1}{3}\left(\frac{2}{3}\right)^3$	$\left(\frac{2}{3}\right)^4$

2 (p. 186, #26)

$$E[X] = \sum_{k=1}^{\infty} k \, P(X=k)$$

$$= P(X=1) + 2P(X=2) + 3P(X=3) + \cdots$$

$$\begin{aligned} = P(X=1) &+ P(X=2) + P(X=3) + P(X=4) + \cdots \\ &+ P(X=2) + P(X=3) + P(X=4) + \cdots \\ &\qquad\quad + P(X=3) + P(X=4) + \cdots \\ &\qquad\qquad\qquad\quad + P(X=4) + \cdots \end{aligned}$$

$$= P(X\geq 1) + P(X \geq 2) + P(X\geq 3) + P(X\geq 4) + \cdots$$

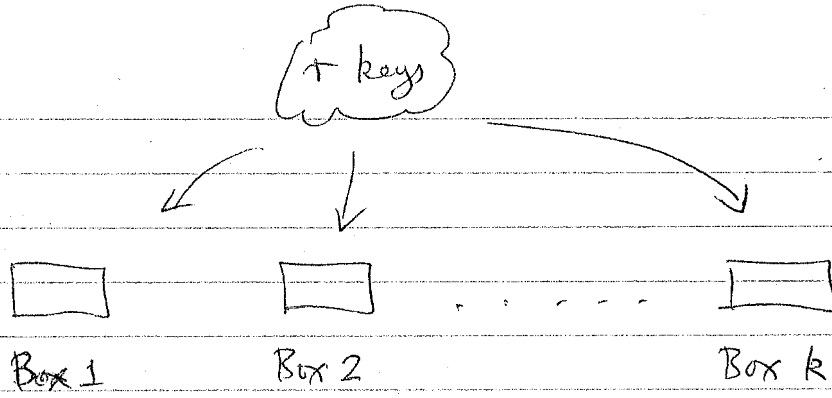
$$= \sum_{k=1}^{\infty} P(X\geq k).$$

3 After the first, each toss is independently a changeover with probability $\frac{1}{2}$. So distribution is binomial with $(n-1, \frac{1}{2})$.

so $P(k \text{ changeovers}) = \binom{n-1}{k}\left(\frac{1}{2}\right)^{k}\left(\frac{1}{2}\right)^{n-1-k}$

$$= \binom{n-1}{k}\left(\frac{1}{2}\right)^{n-1}$$

4.



Let X_i = # keys put in box i

then number of collisions in box i is

$$C_i = \begin{cases} X_i - 1 & \text{if } X_i \geq 1 \\ 0 & \text{if } X_i = 0. \end{cases}$$

so $$E[C_i] = \sum_{k=1}^{r} (k-1)\, P(X_i = k)$$

$$= \sum_{k=1}^{r} k\, P(X_i = k) - \sum_{k=1}^{r} P(X_i = k)$$

$$= \sum_{k=0}^{r} k\, P(X_i = k) - \sum_{k=0}^{r} P(X_i = k) + P(X_i = 0)$$

$$= E[X_i] - 1 + P(X_i = 0).$$

Now X_i is binomial : $X_i \sim \text{Bin}(r, p_i)$

(since each key is independently selected to go into box i with probability p_i).

hence $$P(X_i = 0) = (1 - p_i)^r$$

$$E[X_i] = r p_i$$

$$\Rightarrow \quad E[C_i] = rp_i - 1 + (1-p_i)^r.$$

Now the total number of collisions is

$$C = C_1 + C_2 + \cdots + C_k.$$

hence

$$E[C] = E[C_1] + \cdots + E[C_k]$$

$$= rp_1 - 1 + (1-p_1)^r + \cdots + rp_k - 1 + (1-p_k)^r$$

$$= r - k + \sum_{i=1}^{k} (1-p_i)^r.$$

$\boxed{5}$. Let N_1 = # tosses until first Heads

N_2 = # tosses until second Heads

⋮

N_r = # tosses until r^{th} Heads

Write

$$N_r = N_r - N_{r-1} + N_{r-1} - N_{r-2} + N_{r-2} - N_{r-3}$$

$$+ \cdots + N_2 - N_1 + N_1$$

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Now $N_2 - N_1$ = # tosses between first and second Heads.

Since tosses are independent $N_2 - N_1$ has the same pdf as N_1. Hence

$$E[N_2 - N_1] = E[N_1]$$

Similarly for all $N_k - N_{k-1}$, hence

$$E[N_r] = r\,E[N_1].$$

Now

$$P(N_1 \geq k) = P(\text{all Tails on first } k-1 \text{ tosses})$$

$$= (1-p)^{k-1}$$

Hence (using result on $\boxed{2}$ above)

$$E[N_1] = \sum_{k \geq 1}^{\infty} P(N_1 \geq k)$$

$$= \sum_{k \geq 1}^{\infty} (1-p)^{k-1}$$

$$= \frac{1}{1-(1-p)} = \frac{1}{p}.$$

$$\Rightarrow E[N_r] = \frac{r}{p}.$$