MTH 581: Sp 08

Assignment #3: SOLUTIONS

1 (p.233)

(#5) since expectation is linear we get

$$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]$$
$$= \sum_{i=1}^{n} a_i \mu$$

so condition is $\sum_{i=1}^{n} a_i = 1$.

(#6) Let X_i = stock gain on day #i.

X_i	$-\frac{1}{8}$	$+\frac{1}{8}$
prob	q	p

$\Rightarrow E[X_i] = \frac{1}{8}(p-q)$.

After n days, gain is

$$X_1 + X_2 + \cdots + X_n$$

$\Rightarrow$ expected gain is

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$
$$= \frac{n}{8}(p-q).$$

(#8) Let X, Y be values of faces of dice. Since X, Y are independent we get

$$E[XY] = E[X]\,E[Y] = \left(\frac{7}{2}\right)^2.$$

2 (p. 236)

(#16) $X \sim \text{Bin}(n, p_X)$, $\quad Y \sim \text{Bin}(m, p_Y)$

$$W = 4X + 6Y$$

$\Rightarrow E[W] = 4E[X] + 6E[Y] = 4np_X + 6mp_Y$

Since X, Y independent, get

$$VAR[W] = VAR[4X] + VAR[6Y] = 16\,VAR[X] + 36\,VAR[Y]$$

$$= 16\,n\,p_X(1-p_X) + 36\,m\,p_Y(1-p_Y).$$

(#20) Let X_k = winnings in k^{th} hand.

pdf of X_k:

X_k	0	k
prob	$1-p$	p

T = total winnings $= X_1 + X_2 + \dots + X_n$.

$\Rightarrow E[T] = \sum_{k=1}^{n} E[X_k]$

Independent $\Rightarrow VAR[T] = \sum_{k=1}^{n} VAR[X_k]$.

Now $E[X_k] = kp$

$E[X_k^2] = k^2 p$

$\Rightarrow VAR[X_k] = k^2p - k^2p^2 = k^2p(1-p)$.

$\Rightarrow E[T] = \sum_{k=1}^{n} kp = \frac{n(n+1)}{2} p$

$VAR[T] = \sum_{k=1}^{n} k^2 p(1-p) = \frac{n(n+1)(2n+1)}{6} p(1-p)$.

$\boxed{3}$ (p.279)

(#1). Exact: Prob $= \left(1 - \frac{1}{3250}\right)^{6000}$

Poisson: $\lambda = np = \frac{6000}{3250}$; prob: $= e^{-\lambda}$

(#3) Let $X = \#$ born on Poisson's birthday

$X \sim Bin(500, p)$ $p = \frac{1}{365}$; $\lambda = np = \frac{500}{365}$

$P(X \le 1) = P(X=0) + P(X=1)$

$= e^{-\lambda} + \lambda e^{-\lambda}$.

(#4) (a) $X =$ # babies with mutation

$$X \sim \mathrm{Bin}(20{,}000,\ p) \qquad p = 10^{-4}$$

$$\Rightarrow \lambda = np = 2$$

$$P(X=3) = \frac{\lambda^3}{3!} e^{-\lambda} = \frac{8}{6} e^{-2}$$

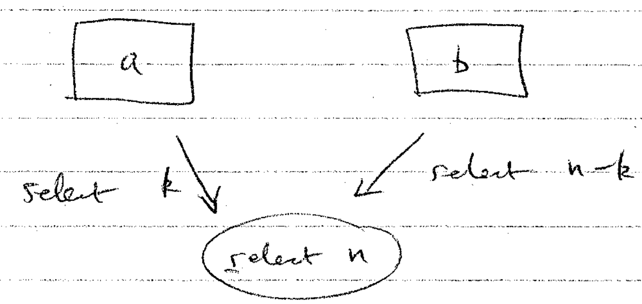
(b)

$$P(X \geq k) = 1 - P(X \leq k-1)$$

$$= 1 - \left(e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!} e^{-\lambda} + \cdots + \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}\right).$$

k	1	2	3	4	5	6
$P(X \geq k)$	0.86	0.59	0.32	0.14	0.05	0.017

$\boxed{14}$



$$\Rightarrow \binom{a+b}{n} = \binom{a}{0}\binom{b}{n} + \binom{a}{1}\binom{b}{n-1} + \cdots + \binom{a}{n}\binom{b}{0}$$

omit terms $\binom{a}{k}$ with $k > a$

and $\binom{b}{k}$ with $k > b$.

$\boxed{5}$ Stirling: $n! \sim n^n e^{-n} \sqrt{2\pi n}$.

$$\Rightarrow \binom{2n}{n} \sim \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi(2n)}}{\left(n^n e^{-n} \sqrt{2\pi n}\right)^2}$$

$$= \frac{2^{2n}\sqrt{4\pi n}}{2\pi n} = \frac{1}{\sqrt{\pi n}}\, 2^{2n}.$$

$\boxed{6}$ Let $X =$ # occurrences of digit 7.

$\Rightarrow X \sim \text{Bin}(n, p)$ $\quad n = 10{,}000$, $\quad p = 0.1$

Use normal approx:

$$X = \sigma Z + \mu$$

$$\mu = E[X] = np = 1000$$

$$\sigma = \text{STD}[X] = \sqrt{np(1-p)} = \sqrt{900} = 30.$$

so $P(X \le 968) = P(30Z + 1000 \le 968)$

$$= P\left(Z \le -\frac{32}{30}\right)$$

$$= P(Z \le -1.07)$$

$$= 0.1423$$

$\boxed{7}$ $X = \#$ Heads in 1000 tosses

Want $P(440 \le X \le k) = 0.5$

~~Now~~ $X \sim Bin(n, p)$ $\quad n = 1000 \quad p = \frac{1}{2}$

$\Rightarrow \mu = np = 500$

$\sigma = \sqrt{np(1-p)} = \sqrt{250} = 15.811$

<u>Normal approx:</u>

$$X = \sigma Z + \mu$$

$$\Rightarrow P(440 \le \sigma Z + \mu \le k) = P\left(-\frac{40}{15.8} \le Z \le \frac{k-500}{15.8}\right)$$

$$= \Phi\left(\frac{k-500}{\sigma}\right) - \Phi\left(-\frac{40}{\sigma}\right)$$

$$= \frac{1}{2}$$

$$\Rightarrow \Phi\left(\frac{k-500}{\sigma}\right) = \frac{1}{2} + \Phi\left(\frac{-40}{\sigma}\right) = \frac{1}{2} + 0.0057 = 0.5057$$

$$\Rightarrow \frac{k-500}{\sigma} = 0.01425$$

$$\Rightarrow k = 500.225$$