MTH 581: Sp 08 (1)

$\boxed{\text{Assignment \#4}}$: SOLUTIONS

$\boxed{1}$

$$N_1 = \min\{n : S_{n-1} = S_n\}$$
$$= \min\{n : X_n = 0\}.$$

so $P(N_1 \geq k) = P(X_1 \neq 0, X_2 \neq 0, \ldots, X_{k-1} \neq 0)$

$= P(X_1 \neq 0)\, P(X_2 \neq 0) \ldots P(X_{k-1} \neq 0)$ ($\because$ independent)

$= (1-r)^{k-1}.$

hence $E[N_1] = \sum_{k=1}^{\infty} P(N_1 \geq k)$

$= \sum_{k=1}^{\infty} (1-r)^{k-1}$

$= \frac{1}{1-(1-r)} = \frac{1}{r}.$

Since N_1 is a stopping time, Wald's theorem implies

$E[S_{N_1}] = E[X]\, E[N_1]$

$= \frac{(p-q)}{r}.$

[2] $N_2 = \min\{n : S_n = S_{n+1}\}$

hence $N_2 = N_1 - 1$, and $S_{N_2} = S_{N_1}$.

so

$$E[N_2] = \frac{1}{r} - 1, \quad E[S_{N_2}] = \frac{p-q}{r}.$$

So Wald's theorem does not hold. This is OK because N_2 requires peeking into the future to determine if the event $\{N_2 = n\}$ has occurred.

[3]. Symmetric random walk:

$$N_3 = \min\{n : S_n = -1\}.$$

From class:

$$P(N_3 = n) = \binom{n-1}{\frac{n-1}{2}} \frac{2}{n+1} \left(\frac{1}{2}\right)^n$$

Stirling $\Rightarrow$ $$\binom{n-1}{\frac{n-1}{2}} \sim \frac{1}{\sqrt{\pi\left(\frac{n-1}{2}\right)}} 2^{n-1}.$$

$$\Rightarrow P(N_3 = n) \sim \frac{1}{\sqrt{\pi\left(\frac{n-1}{2}\right)}} 2^{n-1} \frac{2}{n+1} 2^{-n}$$

$$= \frac{1}{(n+1)\sqrt{\pi\left(\frac{n-1}{2}\right)}} \sim \sqrt{\frac{2}{\pi}}\, n^{-3/2} \quad \text{as } n \to \infty$$

Now we see

$$E[N_3] = \sum_{n\geq 1}^{\infty} n\, P(N_3 = n)$$

for large n this is

$$\sim n \sqrt{\frac{2}{\pi}}\, n^{-3/2} = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}$$

This series diverges!!

hence $E[N_3] = +\infty$.

$\boxed{4}$ Since $N_4 > N_3 \Rightarrow E[N_4] = +\infty$ also.

$\boxed{5}$ $N_1 =$ # tosses until first Head

$N_2 =$ # tosses until ~~first Head~~ two consecutive Heads.

Easy to see $E[N_1] = 2$.

Suppose $N_1 = k$. Then

$$N_2 = \begin{cases} k+1 & \text{with prob } \frac{1}{2} \\ k+1+N_2' & \text{with prob } \frac{1}{2} \end{cases}$$

where N_2' has the same pdf as N_2, since the "clock starts over" once we get Tails.

hence if $N_1 = k$, then

$$E[N_2 \mid N_1 = k] = (k+1)\tfrac{1}{2} + (k+1+E[N_2'])\tfrac{1}{2}$$

$$= k+1 + \tfrac{1}{2}E[N_2]$$

$$= N_1 + 1 + \tfrac{1}{2}E[N_2]$$

Since this is true for every N_1 we get

$$E[N_2] = E[N_1] + 1 + \tfrac{1}{2}E[N_2]$$

$$\Rightarrow \quad E[N_2] = 2(E[N_1]+1)$$

$$= 2(2+1)$$

$$= 6.$$