MTH 581: Sp 08

Assignment #9: $\boxed{\text{SOLUTIONS}}$

$\boxed{1}$ Conditioning on X_m:

$$E[X_m X_n \mid X_m = k]$$

$$= \cancel{k}\, k \; E[X_n \mid X_m = k]$$

$$= k^2 \, E[X_{n-m}]$$

$$= k^2 \mu^{n-m}$$

$$\Rightarrow E[X_m X_n \mid X_m] = \mu^{n-m} X_m^2.$$

$$\Rightarrow \quad E[X_m X_n] = \mu^{n-m} E[X_m^2].$$

$\boxed{2}$ a) $P(N(1) = 0) = e^{-\lambda t} = e^{-2}$

b) $E[T_4] = \frac{4}{\lambda} = \frac{4}{2} = 2$

c) $P(N(2) \geq 2) = 1 - P(N(2) = 0) - P(N(2) = 1)$

$$= 1 - e^{-\lambda t} - (\lambda t) e^{-\lambda t}$$

$$= 1 - e^{-4} - 4e^{-4}.$$

3 $\lambda = 3$ per minute.

$N(t) = \#$ arrivals up to time t.

$X(t) = \#$ recorded pulses up to time t.

$$P(X(t)=0) = \frac{2}{3} P(N(t)=0) + \frac{1}{3} P(N(t)>0)$$

$$= \frac{2}{3} e^{-\lambda t} + \frac{1}{3}(1-e^{-\lambda t})$$

$$= \frac{1}{3}(1+e^{-\lambda t})$$

$$E[X(t) \mid N(t)=k] = \frac{2}{3} k$$

$$\Rightarrow \quad E[X(t) \mid N(t)] = \frac{2}{3} N(t)$$

$$\Rightarrow \quad E[X(t)] = \frac{2}{3} E[N(t)] = \frac{2}{3} \lambda t.$$

4 a) Condition on $N(t) = n$.

$\Rightarrow$ arrivals occur at random times in $[0,t]$, uniformly distributed.

Let W_j = waiting time of j^{th} passenger

Then $W_j \sim U[0,t]$.

$$\Rightarrow \quad E[X \mid N(t)=n] = E\left[\sum_{j=1}^{n} W_j \;\middle|\; N(t)=n\right]$$

$$= n\frac{t}{2}$$

$$\Rightarrow \quad E[X \mid N(t)] = \frac{t}{2} N(t)$$

b) $E[X] = \frac{t}{2} E[N(t)] = \frac{t}{2} \lambda t = \frac{\lambda}{2} t^2.$

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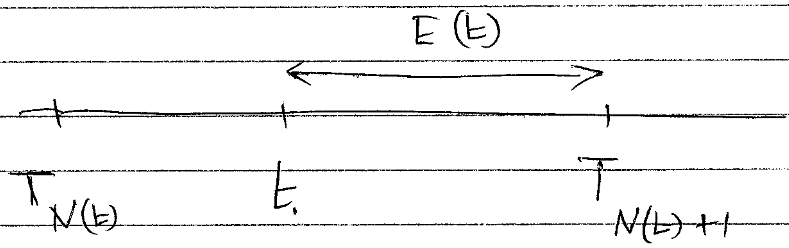
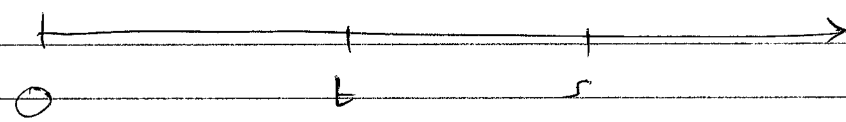
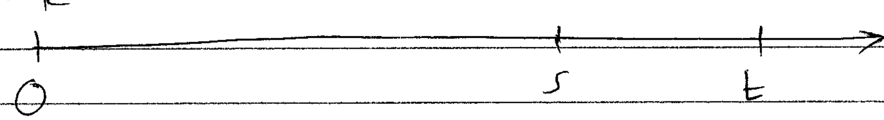


$$P(E(t) > x \mid T_1 = s) = ?$$



If $t < s$, then

$$P(E(t) > x \mid T_1 = s) = \begin{cases} 1 & \text{if } x < s-t \\ 0 & \text{if } x > s-t \end{cases}$$



If $t > s$, then

$$P(E(t) > x \mid T_1 = s) = P(E(t-s) > x)$$

clock starts over at time s.

(4)

Under conditioning,

$$P(E(t)>x) = \int_0^\infty P(E(t)>x \mid T_1 = s)\, \lambda e^{-\lambda s}\, ds$$

$$= \int_0^t P(E(t-s)>x)\, \lambda e^{-\lambda s}\, ds$$

$$+ \int_{t+x}^\infty \lambda e^{-\lambda s}\, ds$$

$$= e^{-\lambda(x+t)} + \int_0^t P(E(t-s)>x)\lambda e^{-\lambda s}\, ds$$

$u = t-s$

$du = -ds$

$$= e^{-\lambda(x+t)} + \int_0^t P(E(u)>x)\, \lambda e^{-\lambda(t-u)}\, du$$

$\frac{d}{dt}$: let $g(t) = P(E(t)>x)$

$$\Rightarrow g'(t) = -\lambda e^{-\lambda(x+t)} + \lambda g(t)$$

$$- \lambda \int_0^t P(E(u)>x)\, \lambda e^{-\lambda(t-u)}\, du$$

$$= -\lambda e^{-\lambda(x+t)} + \lambda g(t) - \lambda\left[g(t) - e^{-\lambda(x+t)}\right]$$

$$= 0$$

$\Rightarrow g(t)$ does not depend on t.

so let $h(x) = P(E(t) > x) = P(E(u) > x)$

$$\Rightarrow h(x) = e^{-\lambda(x+t)} + h(x) \int_0^t \lambda e^{-\lambda t} e^{\lambda u} \, du$$

$$= e^{-\lambda(x+t)} + h(x) e^{-\lambda t} (e^{\lambda t} - 1)$$

$$= e^{-\lambda(x+t)} + h(x) - h(x) e^{-\lambda t}$$

$$\Rightarrow h(x) e^{-\lambda t} = e^{-\lambda(x+t)}$$

$$\Rightarrow h(x) = e^{-\lambda x}$$

$$\Rightarrow \boxed{P(E(t) > x) = e^{-\lambda x}}$$