MTH 581: Sp 09

Assignment #2: $\boxed{\text{SOLUTIONS}}$

$\boxed{1}$ (p.150)

(5) $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{2}$, $P(D) = \frac{1}{4}$, $P(E) = \frac{3}{8}$.

(a) $P(A \cap B) = \frac{1}{4} = P(A)P(B) \Rightarrow$ indep.

$P(A \cap D) = \frac{1}{8} = P(A)P(D) \Rightarrow$ indep.

$P(A \cap E) = \frac{1}{8} \neq P(A)P(E) \Rightarrow$ dependent

$P(D \cap E) = 0 \neq P(D)P(E) \Rightarrow$ dependent.

(b) $P(A \cap B) = P(A)P(B)$ ✓

$P(A \cap C) = P(A)P(C)$ ✓

$P(B \cap C) = P(B)P(C)$ ✓ $\Rightarrow$ independent

$P(A \cap B \cap C) = P(A)P(B)P(C)$ ✓

$P(A \cap B) = P(A)P(B)$ ✓

$P(A \cap D) = P(A)P(D)$ ✓

$P(B \cap D) = \frac{1}{8} = P(B)P(D)$ ✓

$P(A \cap B \cap D) = 0 \Rightarrow$ dependent

$P(D \cap E) = 0 \Rightarrow$ dependent.

(6) # ways to get sum = 12 is 10

246 426 462 444
264 624 642
228 282 822

$\Rightarrow P(\text{get 2 twice} \mid \text{sum} = 12) = \frac{3}{10}$.

(14) $\frac{1}{2} = \frac{P(A \cap B)}{P(B)}$, $P(B) = \frac{3}{4}$

$\Rightarrow P(A \cap B) = \frac{3}{8}$.

22. A: coin comes up Heads 6 times in a row

B: coin is 2-Headed.

Know: $P(B) = \frac{1}{65}$

$$P(A|B) = 1 \qquad P(A|B^c) = \left(\tfrac{1}{2}\right)^6$$

$$\Rightarrow \quad P(B|A) = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)} = \frac{\frac{1}{65}}{\frac{1}{65} + \frac{64}{65}\,\frac{1}{2^6}} = \frac{1}{2}$$

39. a) $P(X \geq 1, Y \leq 0) = P(1,0) + P(1,-1) + P(2,0) + P(2,-1)$

$$= \frac{1}{12} + \frac{1}{36} = \frac{4}{39} = \frac{1}{9}.$$

b) $$P(Y \leq 0 \mid X = 2) = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{12} + \frac{1}{6}} = \frac{1}{1+1+2} = \frac{1}{4}$$

c) NO: e.g. $P(X=-1, Y=-1) = 0 \neq P(X=-1)\,P(Y=-1)$

d) $\mathrm{Ran}\, Z = \mathrm{Ran}(XY) = \{-2, -1, 0, 1, 2, 4\}$

Z	-2	-1	0	1	2	4
prob	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$.

← uniform!

52. C_i: coin is in box #i

F_i: you ~~find~~ ~~coin~~ search box #i and find the coin

Given: $P(C_i) = p_i \qquad P(F_i \mid C_i) = a_i \qquad P(F_i \mid C_j) = 0.$

$$p = P(C_j \mid F_i^c) = \frac{P(F_i^c \mid C_j)\,P(C_j)}{P(F_i^c \mid C_j)\,P(C_j) + \sum_{k \neq j} P(F_i^c \mid C_k)\,P(C_k)}$$

If $j \neq i \Rightarrow p = \dfrac{1 \cdot p_j}{p_j + \sum_{k \neq j, i} p_k + (1-a_i)\, p_i}$

$$= \frac{p_j}{1 - p_i a_i}$$

If $j = i \Rightarrow p = \dfrac{(1-a_i)\, p_i}{(1-a_i)\, p_i + \sum_{k \neq i} p_i}$

$$= \frac{(1-a_i)\, p_i}{1 - a_i p_i}$$

$\boxed{2}$ (p.247)

5. X = winnings

X	-1	1
prob.	$\frac{20}{38}$	$\frac{18}{38}$

$$E[X] = -\frac{20}{38} + \frac{18}{38} = -\frac{1}{19}.$$

14. $X_j = \begin{cases} 1 & \text{if } j^{\text{th}} \text{ person gets own hat} \\ 0 & \text{else.} \end{cases}$

$$P(X_j = 1) = \frac{(n-1)!}{n!} = \frac{1}{n} \Rightarrow E[X_j] = \frac{1}{n}.$$

$j \neq k$: $P(X_j = 1, X_k = 1) = \dfrac{(n-2)!}{n!} = \dfrac{1}{n(n-1)}$.

$$\Rightarrow E[X_j X_k] = \frac{1}{n(n-1)} \neq E[X_j]\, E[X_k]$$

$\Rightarrow$ dependent.

26) Assume $S \geq C$

(a) Walk below line L $\Rightarrow$ more stars than circles left

$\Rightarrow$ guess star at next step

So if walk moves up then you guessed right.

Similarly if walk above line L.

Let $A = \#$ steps above line L

$B = \#$ steps below line L

Then $A + B = S + C$.

Also $\#$ correct guesses above $L = \frac{1}{2} A$

$\#$ ——"—— below $L = \frac{1}{2}(B) + S - C$

$\Rightarrow$ $\#$ correct guesses $= \frac{1}{2}A + \frac{1}{2}B + S - C = \frac{1}{2}(S+C) + S - C$

$\geq \frac{1}{2}(S+C) + \frac{1}{2}(S-C)$

$= S.$