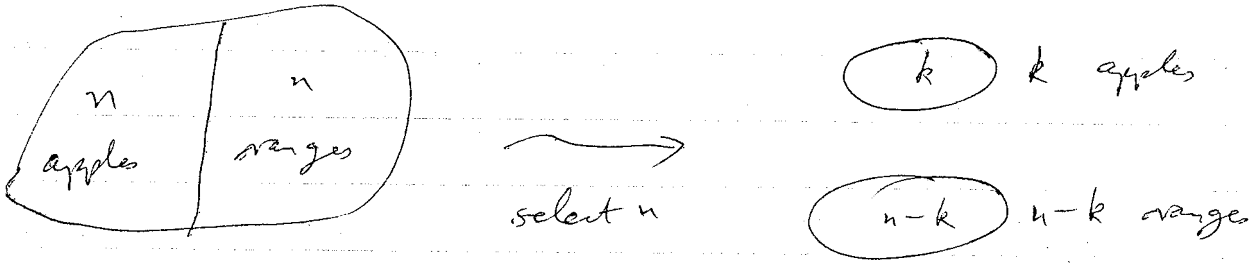
MTH 581

Assignment #8

SOLUTIONS

1 a).



$$\Rightarrow \# \text{ of ways} = \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

$k = \#$ apples chosen.

b)
$$p_{00}^{(2m)} = \sum_{k=0}^{m} \binom{2m}{2k}\binom{2k}{k}\binom{2m-2k}{m-k}\left(\frac{1}{4}\right)^{2m}$$

$$= \sum_{k=0}^{m} \frac{(2m)!\,(2k)!\,(2m-2k)!}{(2m-2k)!\,(2k)!\,(k!)^2\,(m-k)!^2}\left(\frac{1}{4}\right)^{2m}$$

$$= \sum_{k=0}^{m} \frac{(2m)!}{(k!)^2\,(m-k)!^2}\left(\frac{1}{4}\right)^{2m}$$

$$= \frac{(2m)!}{(m!)^2}\sum_{k=0}^{m} \frac{(m!)^2}{(k!)^2\,((m-k)!)^2}\left(\frac{1}{4}\right)^{2m}$$

$$= \binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m}{m-k}\left(\frac{1}{4}\right)^{2m}$$

$$= \binom{2m}{m}^2\left(\frac{1}{4}\right)^{2m}.$$

c) Stirling:

$$\binom{2m}{m} = \frac{(2m)!}{(m!)^2} \simeq \frac{(2m)^{2m}\sqrt{2\pi(2m)}\,e^{-2m}}{\left(m^m\sqrt{2\pi m}\,e^{-m}\right)^2}$$

$$= \frac{1}{\sqrt{\pi m}}\,2^{2m}$$

$$\Rightarrow \quad p_{00}^{(2m)} \simeq \left(\frac{1}{\sqrt{\pi m}}\right)^2 \quad \text{as}$$

$$= \frac{1}{\pi m} \quad \text{as} \quad m \to \infty.$$

d) Since $\sum_{m=1}^{\infty} \frac{1}{m} = \infty \Rightarrow$ walk is persistent.

$\boxed{2}$. $\boxed{\text{Irreducible}}$

Two states (i,k) and (j,ℓ) in $S \times S$.
Want to show that

$$P_{(i,k),(j,\ell)}^{(m)} > 0 \quad \text{for some } m.$$

Now $P_{(i,k)(j,\ell)} = P_{ij}\,P_{k\ell}$

hence

$$P_{(i,k)(j,\ell)}^{(m)} = P_{ij}^{(m)}\,P_{k\ell}^{(m)} \quad \text{all } m \geq 1.$$

Since X, Y are irreducible and aperiodic by assumption, there exist integers n_1 and n_2 such that

$$P_{ij}^{(m)} > 0 \qquad \text{all} \quad m \geq n_1$$

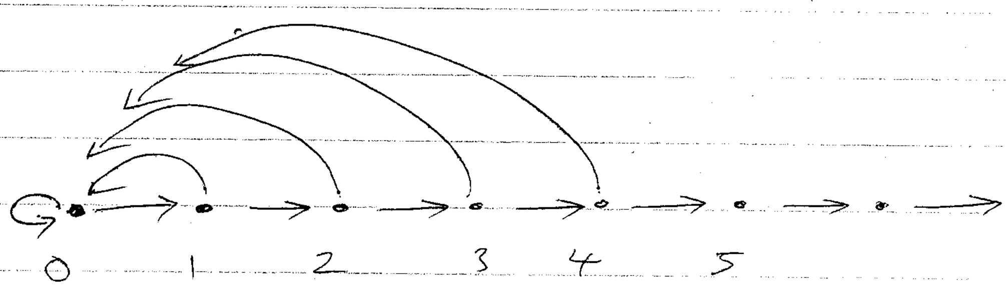
$$P_{kl}^{(m)} > 0 \qquad \text{all} \quad m \geq n_2$$

So take $N = \max(n_1, n_2)$. Then

$$P_{(i,k),(j,l)}^{(m)} > 0 \qquad \text{all} \quad m \geq N.$$

Hence Z is irreducible. Also taking $(i,k) = (j,l)$ shows that Z is aperiodic.

$\boxed{3}$



$$P_{i,i+1} = a_i, \qquad P_{i,0} = 1 - a_i$$

a) <u>Persistent</u>. Clearly chain is irreducible.

$$f_{00}^{(n)} = P(\text{return to } 0 \text{ for first time after } n \text{ steps} \mid \text{start at } 0)$$

$$= P(0 \to 1 \to 2 \to 3 \to \cdots \to n-1 \to 0)$$

$$= a_0 a_1 a_2 \cdots a_{n-2} (1 - a_{n-1})$$

$$= b_{n-1} - b_n$$

Hence $$f_{00} = \lim_{N \to \infty} \sum_{n=1}^{N} f_{00}^{(n)}$$

$$= \lim_{N \to \infty} (b_0 - b_N)$$

$$= 1 - \lim_{N \to \infty} b_N$$

$\Rightarrow$ chain is persistant

$\iff f_{00} = 1$

$\iff \lim_{N \to \infty} b_N = 0.$

(b) <u>Non-null persistent.</u>

Look for stationary distribution:

$$w_j = \sum_i w_i p_{ij}$$

$j \geq 1$: $\quad w_j = a_{j-1} w_{j-1}$

$$\Rightarrow \quad w_j = a_{j-1} a_{j-2} \cdots a_0 w_0$$

$$= b_j w_0 \quad \text{for } j \geq 1.$$

<u>Normalize</u>: $\quad \sum_j w_j = w_0 + \sum_{j \geq 1} b_j w_0$

$$= w_0 \left(1 + \sum_{j=1}^{\infty} b_j \right).$$

Want $\sum_j w_j = 1 \Rightarrow$ choose w_0 to make this true.

This is possible if and only if $\sum_{j=1}^{\infty} b_j < \infty$.

If so, then

$$w_0 = \left(1 + \sum_{j=1}^{\infty} b_j\right)^{-1}$$

$$w_j = b_j \left(1 + \sum_{j=1}^{\infty} b_j\right)^{-1} \quad (j \geq 1)$$

c) $a_i = 1 - A i^{-\beta}$.

$\boxed{\text{Transient}}$ from part (a) $\Leftrightarrow$ $b_i \not\to 0$ as $i \to \infty$

$$b_N = a_0 a_1 a_2 \ldots a_{N-1}$$

$$\Rightarrow \log b_N = \sum_{j=0}^{N-1} \log (a_j)$$

for large j, $\log (a_j) = \log \left[1 - \frac{A}{j^\beta}\right]$

$$\simeq - \frac{A}{j^\beta}.$$

$$\Rightarrow \log b_N \simeq \sum_j - \frac{A}{j^\beta}$$

This converges if and only if $\beta > 1$.

Hence chain is transient if $\beta > 1$.

d) Condition of is $\sum_i b_i < \infty$

Now if $\beta < 1$

$$\Rightarrow \log b_N = \sum_j \frac{-A}{j^\beta} \simeq \frac{-A}{\beta} N^{1-\beta},$$

as $N \to \infty$

so $$\sum b_n \simeq \sum e^{-\frac{A}{\beta} n^{1-\beta}}$$

$$< \infty \quad \text{for all } \beta < 1.$$

$\Rightarrow$ chain is non-null persistent