MTH 581

Practice Problems for Final

$\boxed{1}$ Let $n-a = 2k$, assume $p = q = \frac{1}{2}$.

$\Rightarrow$ ~~Thus~~ $P($ first return to 0 at $2k$ steps $|$ start at $0)$

$= \frac{1}{2} P($ first visit to 0 at $2k-1$ steps $|$ start at $+1)$

$+ \frac{1}{2} P($ —— " —— $|$ — " — $-1)$

$= P($ first visit to -1 at $2k-1$ $|$ start at $0)$

$\uparrow$ $n = 2k-1$, $a = 1$, $p = q = \frac{1}{2}$

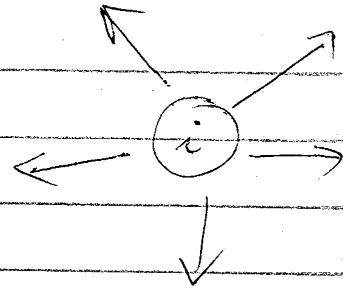
$$= \binom{2k-2}{k-1} \frac{2}{2k} \left(\frac{1}{2}\right)^{2k-1}$$

$$= \frac{1}{k} 2^{-(2k-1)} \binom{2k-2}{k-1}.$$

Stirling $\Rightarrow$ $$\binom{2k-2}{k-1} = \frac{(2k-2)!}{(k-1)!} \simeq \frac{1}{\sqrt{\pi(k-1)}} \cdot 2^{2k-2}$$

$$\Rightarrow \text{Prob} \simeq \frac{1}{k} 2^{-1} \cdot \frac{1}{\sqrt{\pi(k-1)}} = \frac{1}{2k\sqrt{\pi(k-1)}}$$

$\boxed{2}$



$L(i)$ = # outward links from i.

$$R(j) = \frac{1-\delta}{n} + \delta \sum_{i=1}^{n} R(i)\, p_{ij}$$

$$p_{ij} = \begin{cases} \frac{1}{L(i)} & \text{if } i \to j \\ 0 & \end{cases}$$

Let T = transition matrix, $T_{ij} = p_{ij}$

Let $s = (1, 1, \ldots, 1)$ (all 1's)

$R = (R(1), \ldots, R(n))$

Then

$$R = \frac{1-\delta}{n} s + \delta \sum_{i=1}^{n} R(i)\, T_{ij}$$

$$\boxed{R = \frac{1-\delta}{n} s + \delta R T}$$

Iterate to find solution;

$$R_0 = \frac{1-\delta}{n} s$$

$$R_1 = R_0 + \delta R_0 T$$

$$= \frac{1-\delta}{n}\left[s + \delta s T \right]$$

$$R_2 = R_0 + \delta R_1 T$$

$$= \frac{1-\delta}{n}\left[s + \delta s T + \delta^2 s T^2 \right]$$

etc.

$$R_{k+1} = \frac{1-\delta}{n}\left[s + \delta s T + \delta^2 s T^2 + \cdots + \delta^{k+1} s T^{k+1} \right]$$

Since $R_{k+1} = R_0 + \delta R_k T$

let $k \to \infty \Rightarrow R_k \to R$

$$\Rightarrow R = \frac{1-\delta}{n}\left[s + \delta s T + \delta^2 s T^2 + \cdots \right]$$

$$= \frac{1-\delta}{n} \sum_{k=0}^{\infty} \delta^k s T^k$$

$$R(j) = \frac{1-\delta}{n} \sum_{k=0}^{\infty} \delta^k \sum_i P_{ij}^{(k)}$$

3 Let N_1 = # plays until first loss

N_2 = # plays until second loss.

$E[N_1] = \frac{1}{q}$

$E[N_2 - N_1] = \frac{1}{q}$

$\Rightarrow E[N_2] = \frac{2}{q}$ = expected duration of play.

Note that

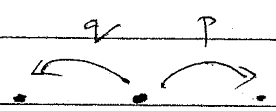
$$X_{N_1} = 1 + 2 + 4 + \cdots + 2^{N_1 - 2} - 2^{N_1 - 1} = -1$$

and

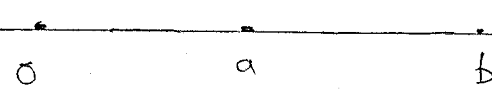
$$X_{N_2} - X_{N_1} = 1 + 2 + 4 + \cdots + 2^{N_2 - N_1 - 2} - 2^{N_2 - N_1 - 1} = -1$$

$\Rightarrow X_{N_2} = -2$ = winnings after you stop.

4



a) $\underline{b > a}$: start at 0



P(reach b before 0 | start at a)

$$= \begin{cases} \dfrac{1 - r^a}{1 - r^b} & r = \dfrac{q}{p} \neq 1 \quad \text{(from class)} \\[2ex] \dfrac{a}{b} & r = 1 \end{cases}$$

b)

P(never reaches 0 | ~~start~~ a 0)

$= \lim_{b\to\infty}$ P(reaches b before 0 | start at a)

$$= \lim_{b\to\infty} \begin{cases} \left(\dfrac{1-r^a}{1-r^b}\right) & r \neq 1 \\ \dfrac{a}{b} & r = 1 \end{cases}$$

$$= \begin{cases} 1 - r^a & r < 1 \iff q < p \\ 0 & r \geq 1 \iff q \geq p \end{cases}$$

c) N = # steps until last visit to 0.

$= \max\{n : X_n = 0\}$.

Now N exists if and only if $q < p$.

(if $q \geq p$ there is no last visit).

Also, must have $\boxed{X_{N+1} = 1}$, otherwise we have $X_{N+1} = -1$

but then since $q < p$ it will revisit 0.

So $\{N = n\} = \{X_n = 0, X_{n+1} = 1, X_m \neq 0 \text{ all } m > n\}$

d) $P(N=n) = P(X_n = 0,\ Y_{n+1} = 1,\ X_m \neq 0 \quad \text{all } m > n)$

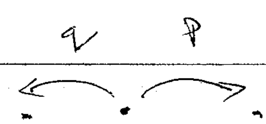
$$= P(X_m \neq 0 \quad \text{all } m>n \mid X_{n+1} = 1) \cdot P(X_n = 0,\ Y_{n+1} = 1)$$

$$= P(\text{walk never reaches } 0 \mid \text{start at } +1) \cdot p \cdot P(X_n = 0).$$

$$= (1-r)\, p \cdot P(X_n = 0).$$

$$\Rightarrow \boxed{P(N=2k) = (p-q) \binom{2k}{k} p^k q^k}$$

5.



$N = \#$ steps until $Y_{n-1} = Y_n$ first time.

Same as coin toss: $N = \#$ tosses until two consecutive Tails

From previous work: $E[N] = \frac{1}{q} + \frac{1}{q^2}$.

Wald's Theorem $\Rightarrow$ $E[X_N] = (p-q)\, E[N]$.