

MTH U581: Spring 2009: Prof. C. King

Assignment 8

Due date: Thursday, April 2.

1. a) Derive the following identity:

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k}$$

[Hint: think about the number of ways to select n pieces of fruit from a basket with n apples and n oranges.]

b) In class we derived the formula

$$p_{00}^{(2m)} = \sum_{k=0}^m \binom{2m}{2k} \binom{2k}{k} \binom{2m-2k}{m-k} \left(\frac{1}{4}\right)^{2m}$$

for the probability that a random symmetric walk on the two-dimensional lattice \mathbf{Z}^2 which starts at the origin will return there after $2m$ steps. Use the result of part (a) to deduce that

$$p_{00}^{(2m)} = \left(\binom{2m}{m} 4^{-m} \right)^2$$

c) Apply Stirling's formula to show that

$$p_{00}^{(2m)} \simeq \frac{1}{\pi m}$$

as $m \rightarrow \infty$.

d) Decide whether the random symmetric walk on \mathbf{Z}^2 is persistent or transient.

2) Suppose that X and Y are independent irreducible aperiodic Markov chains with the same state space S and the same transition matrix P . Let $Z = (X, Y)$ be the chain with state space $S \times S$ and transition matrix

$$P\left(Z_1 = (j, l) \mid Z_0 = (i, k)\right) = P(X_1 = j \mid X_0 = i) P(Y_1 = l \mid Y_0 = k)$$

Show that Z is also irreducible and aperiodic.

3) Consider a Markov chain on the set $S = \{0, 1, 2, \dots\}$ with transition probabilities

$$p_{i,i+1} = a_i, \quad p_{i,0} = 1 - a_i$$

for $i \geq 0$, where $\{a_i \mid i \geq 0\}$ is a sequence of constants which satisfy $0 < a_i < 1$ for all i . Let $b_0 = 1$, $b_i = a_0 a_1 \cdots a_{i-1}$ for $i \geq 1$. Show that the chain is

- (a) persistent if and only if $b_i \rightarrow 0$ as $i \rightarrow \infty$
- (b) non-null persistent if and only if $\sum_i b_i < \infty$,

and write down the stationary distribution if the latter condition holds.

Let A and β be positive constants and suppose that $a_i = 1 - Ai^{-\beta}$ for all large values of i . Show that the chain is

- (c) transient if $\beta > 1$
- (d) non-null persistent if $\beta < 1$.