

## MTH U581 : PRACTICE PROBLEMS FOR FINAL

1). Consider a random walk on the integers with probability  $p$  to step right and probability  $q$  to step left. Let  $N$  be the number of steps until the walk first reaches the point  $-a$ , starting at 0. In class we derived the following formula (assuming  $n - a$  is even):

$$P(N = n) = \binom{n-1}{(n-a)/2} \frac{2a}{n+a} p^{(n-a)/2} q^{(n+a)/2}$$

Use this formula to derive the probability that the walk first returns to 0 after exactly  $2k$  steps, given that the walk started at 0. Then use Stirling's formula to show that for large  $k$ , this probability is approximately

$$\frac{1}{2k\sqrt{\pi(k-1)}}$$

2). The Google page-rank algorithm assigns a probability distribution  $R(1), \dots, R(n)$  to a collection of  $n$  webpages. The webpages are connected by directed links. For each page  $i$ , let  $L(i)$  be the number of outward directed links starting at  $i$  (assume that every webpage has at least one outward directed link so  $L(i) \geq 1$  for all  $i$ ). Then the probability distribution  $R$  is the solution of the equation

$$R(j) = \frac{(1-\delta)}{n} + \delta \sum_{i=1}^n R(i)p_{ij}$$

where the transition matrix  $P$  has entries

$$p_{ij} = \begin{cases} L(i)^{-1} & \text{if there is a directed link from } i \text{ to } j \\ 0 & \text{else} \end{cases}$$

and where  $\delta < 1$  is a positive 'damping factor'.

As usual let  $p_{ij}^{(n)}$  denote the entries of the matrix  $P^n$ . Show that  $R$  satisfies the following equation:

$$R(j) = \frac{(1-\delta)}{n} \sum_{k=0}^{\infty} \delta^k \sum_{i=1}^n p_{ij}^{(k)}$$

3). In a game of ‘double or nothing’ your winnings after  $n$  rounds are

$$X_n = Y_1 + 2Y_2 + 4Y_3 + \cdots + 2^{n-1}Y_n$$

where you win in the  $i^{\text{th}}$  round if  $Y_i = 1$  and you lose if  $Y_i = -1$ . Assume that  $P(Y_i = 1) = p$  and  $P(Y_i = -1) = q$  with  $p+q = 1$ . You decide to stop after you lose for the second time. Find the expected duration of play and find your expected winnings after you stop.

4). Consider a random walk on the integers, starting at position  $a > 0$

$$X_n = a + Y_1 + \cdots + Y_n$$

with probabilities  $p, q$  of right and left steps respectively, so

$$P(Y = 1) = p, \quad P(Y = -1) = q$$

Suppose that  $p > q$ .

a) Let  $b > a$ . Find the probability that the walk reaches the point  $b$  before it reaches 0 (we did this in class).

b) Find the probability that the walk never reaches the point 0.

c) Let  $N$  be the number of steps until the walk visits the point 0 for the last time (note that  $N$  is NOT a stopping time!). So  $N = \max\{n : X_n = 0\}$ . Explain why the event  $\{N = n\}$  is the same as the event

$$\{X_n = 0, \quad Y_{n+1} = 1, \quad X_m \neq 0 \text{ for } m > n\}$$

d) Find  $P(N = n)$ .

5). Let  $X_n = a + Y_1 + \cdots + Y_n$  be a random walk on the integers, with probabilities  $p, q$  of right and left steps respectively. Let  $N$  be the number of steps until  $Y_{n-1} = Y_n$  for the first time. Find  $E[N]$  and  $E[X_N]$ .