# MTH U581 : PRACTICE PROBLEMS FOR TEST 2

**1).** Every vehicle on a road is either a car or a truck. Three our of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

[Hint: set this up as a two state Markov chain and find the stationary distribution].

*Answer:* 4/19

**2).** A rat is put in a box with five doors which are arranged in a circle. The doors are labeled #1 through #5 in clockwise order. Door #1 leads to freedom, the other four doors are locked. At every time unit the rat chooses one of the doors, and attempts to pass through it. If it fails then it makes another attempt at the next time unit. It will not attempt the same door twice in a row. It is twice as likely to choose one of the neighboring doors for the next attempt as to choose one of the other two doors.

So for example, if it chooses Door #3 at one attempt, then at the next attempt it will choose either Door #2 or Door #4 with probability 1/3, and the others with probability 1/6.

Model this by a five state Markov chain, where the state is the most recent choice made by the rat. Write down the transition matrix. Find the expected number of steps until the rat escapes, for each of the four possible initial choices.

*Answer:* for doors #2,3,4,5: $42/11, 48/11, 48/11, 42/11$

**3).** Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 0.5 & 0.3 & 0 & 0 & 0.2 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.4 & 0.4 & 0.2 & 0 \\ 0.3 & 0 & 0.2 & 0 & 0.5 \\ 0.5 & 0.2 & 0 & 0 & 0.3 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

a). Find and classify the equivalence classes of the states.

*Answer:* $\{1, 2, 5\}$ ergodic, $\{3, 4\}$ transient

b). Find a stationary distribution for the chain. Is it unique?

*Answer:* $w = (1/3, 1/3, 0, 0, 1/3)$, yes.

c). Compute the expected number of steps needed to first return to state 1, conditioned on starting in state 1.

*Answer:* 3

d). Compute the expected number of steps needed to first reach any of the states $\{1, 2, 5\}$, conditioned on starting in state 3.

*Answer:* 15/7

**4).** An unpopular math professor gives three types of exams, namely easy (E), medium (M) and hard (H). The class either does well or does badly on an exam. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type H.

The probability that the class does well on each exam type is as follows:

$$p_E = 0.3, \quad p_M = 0.6, \quad p_H = 0.9$$

Find the long-run proportion of exams of types E, M, H.

*Answer:* $w = (3/13, 3/13, 7/13)$

**5).** Suppose that a Markov chain has $M$ states. Show that for any two states $i, j$ of the chain, either $p_{ij}^{(n)} = 0$ for all $n$, or else $p_{ij}^{(n)} > 0$ for some $n \leq M$.

*Answer:* either $p_{ij}^{(n)} = 0$ for all $n$, or for some $n$ $p_{ij}^{(n)} > 0$. If $n \leq M$ we are done. If not, then there is some sequence of states $i = i_1, i_2, \dots, i_n = j$ such that

$$p_{i_1,i_2} p_{i_2,i_3} \cdots p_{i_{n-1},i_n} > 0$$

Since $n > M$ there must be repeated states in this sequence. If the state $k$ is repeated in the sequence, then there is a loop from $k$ back to itself. We can remove this loop and the product is still positive. By removing all loops we end up with a sequence of length at most $M$.

**6).** Consider the following transition probability matrix for a Markov chain on 3 states:

$$P = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the fundamental matrix $N$. Suppose the chain is twice as likely to start in state 1 as in state 2 (the states are labeled 1,2,3 in the order written for the matrix). Find the expected time until the chain is absorbed in state 3.

*Answer:* 85/12