MTH 581 : Sp 09

$\boxed{\text{Test 1}}$ Practice Problems.

(1) Label letters 1, 2, 3, ..., 8

Prob. to get CALCULUS = $\dfrac{\#\text{ ways to get it}}{\text{total }\#\text{ ways}} = \dfrac{2^3}{8!}$

(2). (a) Prob. $= \dfrac{\#\text{ ways in different boxes}}{\text{total }\#\text{ ways}} = \dfrac{n!}{n^n}$

(b) Prob. $= \dfrac{\#\text{ ways}}{\text{total}} = \dfrac{n\binom{n}{2}(n-1)!}{n^n}$

$n = \#$ choices of empty box
$\binom{n}{2} = \#$ choices for pair of ~~balls~~
$(n-1)! = \#$ ways to put $(n-1)$ keys in $(n-1)$ boxes.

(3). (a) Prob. $= \dfrac{2n(2n-2)(2n-4)\cdots(2n-4r+2)}{(2n)(2n-1)(2n-2)\cdots(2n-2r+1)}$.

(b) Lay out $2r$ chosen shoes:

$\binom{2r}{2} \rightarrow \#$ ways to choose positions for the equal pair

$2n \rightarrow \#$ ways to select a pair to put there

$(2n-2)(2n-4)\cdots(2n-4r+4) \rightarrow \#$ ways to select remaining $2r-2$ shoes

$\Rightarrow$ Prob. $\dfrac{\binom{2r}{2}(2n)(2n-2)(2n-4)\cdots(2n-4r+4)}{(2n)(2n-1)(2n-2)\cdots(2n-2r+1)}$

(C) $\binom{2r}{4}$ → # ways to select 4 positions for the two complete pairs

$\binom{n}{2}$ → # ways to select 2 complete pairs

$4!$ → # ways to put 2 complete pairs in 4 positions

$(2n-4)(2n-6)\cdots(2n-4r+6)$ → # ways to select remaining $2r-4$ shoes (no pairs)

$$\Rightarrow \text{Prob} = \frac{\binom{2r}{4}\binom{n}{2}\,4!\,(2n-4)(2n-6)\cdots(2n-4r+6)}{(2n)(2n-1)\cdots(2n-2r+1)}$$

Toss coin n times

4 A: Head on toss # ~~k~~ j

B: exactly k Heads

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\left(\frac{1}{2}\right)\binom{n-1}{k-1}\left(\frac{1}{2}\right)^{k-1}}{\binom{n}{k}\frac{1}{2}^{k}}$$

$$= \frac{k}{n}.$$

5 $P(B^c) - P(B^c|A) = 1 - P(B) - 1 + P(B|A)$

$= P(B|A) - P(B)$

> 0

6 $X = \begin{cases} 0 & \text{if } \cancel{\text{first toss}} \text{ Tails} \\ 1 & \text{———''———} \text{ Heads.} \end{cases}$

N = # tosses until at least one Head & one Tail

$E[N|X=0] = 1 + E[\#\text{ tosses to first Head}]$

$= 1 + \frac{1}{p}$

$E[N|X=1] = 1 + E[\#\text{ tosses to first Tail}]$

$= 1 + \frac{1}{1-p}$

$\Rightarrow E[N] = \left(1+\frac{1}{p}\right)(1-p) + \left(1+\frac{1}{1-p}\right)p$

$= 1 + \frac{1-p}{p} + \frac{p}{1-p}$

7. $X = \begin{cases} 0 & \text{if coin } 1 \\ 1 & \text{— " — } 2 \\ 2 & \text{— " — } 3 \end{cases}$

$N = \#$ Heads in 10 tosses

$$P(N=n \mid X=0) = \binom{10}{n}(0.3)^n(0.7)^{10-n}$$

$$P(N=n \mid X=1) = \binom{10}{n}(0.5)^n(0.5)^{10-n}$$

$$P(N=n \mid X=2) = \binom{10}{n}(0.7)^n(0.3)^{10-n}$$

$$\Rightarrow P(N=n) = \frac{1}{3}\binom{10}{n}\left[(0.3)^n(.7)^{10-n} + (.5)^{10} + (.3)^{10-n}(.7)^n\right].$$

8. $Z = 2X + 3Y - 4$

$$g_X(t) = E[e^{tX}] = \frac{1}{4}(1 + e^t + e^{2t} + e^{3t})$$

$$g_Y(t) = g_X(t)$$

$$g_Z(t) = E\left[e^{(2X+3Y-4)t}\right]$$

$$= g_X(2t) \cdot g_Y(3t) \cdot e^{-4t}$$

$$= \frac{1}{16} e^{-4t}(1 + e^{2t} + e^{4t} + e^{6t}).$$

$$(1 + e^{3t} + e^{6t} + e^{9t}).$$