1 Notes on Arbitrage Theorem

Here I describe the basic mathematical result underlying the Arbitrage Theorem. To facilitate visualization the result is stated in two dimensions.

Let $\mathbf{r}(1), \mathbf{r}(2), \mathbf{r}(3)$ be 3 vectors in the plane. So picture them as vectors starting at the origin, and sticking out into the plane. Pose the following question: is there a line L passing through the origin, so that all the 3 vectors lie on one side of L? There are two possible answers, either Yes or No.

1.1 Yes

If the answer is Yes, let $\mathbf{x} = (x_1, x_2)$ be the unit vector orthogonal to L which points on the same side as the vectors $\mathbf{r}(1), \mathbf{r}(2), \mathbf{r}(3)$. Since $\mathbf{x}$ and $\mathbf{r}(1)$ are on the same side of L, the angle between these vectors is less than $\pi/2$, and hence their dot product is positive, that is

$$\mathbf{x} \cdot \mathbf{r}(1) > 0$$

The same is true for the other vectors $\mathbf{r}(2), \mathbf{r}(3)$, hence we have

$$\mathbf{x} \cdot \mathbf{r}(j) > 0 \quad \text{for all} \quad j = 1, 2, 3 \tag{1}$$

1.2 No

Now suppose the answer is No, so that there is no such line L. I claim that this means the following: there must be positive numbers c_1, c_2, c_3 such that

$$c_1\mathbf{r}(1) + c_2\mathbf{r}(2) + c_3\mathbf{r}(3) = 0 \tag{2}$$

Geometrically, imagine the vectors as strings tied down at the origin, and c_j as the tension of the j^{th} string. Then this relation means that you can choose the tensions of the strings so that the net force is zero at the origin.

To see that the relation (2) must be true, we will assume that it is *false* and then derive a contradiction. Let R be the cone-shaped region contained between the two vectors $\mathbf{r}(1)$ and $\mathbf{r}(2)$. Every vector in this region R can be written as $c_1\mathbf{r}(1)+c_2\mathbf{r}(2)$ for some *positive* numbers c_1 and c_2. Consider the following: does the vector $-\mathbf{r}(3)$ lie in R? If it does, then as observed we can write

$$-\mathbf{r}(3) = c_1\mathbf{r}(1) + c_2\mathbf{r}(2) \tag{3}$$

for some positive numbers c_1, c_2. But we are assuming that (2) is false, and so (3) cannot hold. Hence under our assumption we must have the vector $-\mathbf{r}(3)$ lying outside the region R. But then there must be a line L which passes through the origin so that R is on one side and $-\mathbf{r}(3)$ is on the other side. But since $-\mathbf{r}(3)$ and $\mathbf{r}(3)$ lie on opposite sides of L, this means that R and $\mathbf{r}(3)$ lie on the same side of L. The region R contains both vectors $\mathbf{r}(1)$ and $\mathbf{r}(2)$, therefore $\mathbf{r}(1)$, $\mathbf{r}(2)$ and $\mathbf{r}(3)$ all lie on the same side of L. And this is the promised contradiction. Hence the conclusion is that (2) must hold for some positive numbers c_1, c_2, c_3.

1.3 Summarize result in two dimensions

To summarize: either the 3 vectors all lie on the same side of some line, in which case (1) holds for some vector $\mathbf{x}$, or the 3 vectors do not lie on the same side of any line, in which (2) holds for some positive numbers c_1, c_2, c_3. So either (1) or (2) holds, but not both.

1.4 Connection with the betting scheme

To make the connection with the betting scheme, there are two wagers, and there are 3 possible outcomes. Write $\mathbf{r}(j) = (r_1(j), r_2(j))$. If you bet one dollar on the first wager, you get the return $r_1(j)$ depending on the outcome j. Similarly for the second wager. Hence betting x_1 on wager 1 and x_2 on wager 2 yields the return

$$x_1 r_1(j) + x_2 r_2(j) = \mathbf{x} \cdot \mathbf{r}(j)$$

when the outcome is j. Hence (1) says that the return is positive for all 3 outcomes, if you bet x_1 on wager 1 and x_2 on wager 2. So this is one possible situation, known as an *arbitrage*, that is a betting scheme which yields a positive return for every possible outcome.

The other situation arises when (2) holds. Since c_1, c_2, c_3 are positive, we can divide across (2) by the sum $c_1 + c_2 + c_3$ and get

$$p_1 \mathbf{r}(1) + p_2 \mathbf{r}(2) + p_3 \mathbf{r}(3) = 0 \tag{4}$$

where now $p_1 + p_2 + p_3 = 1$. Of course this is really two equations, namely

$$p_1 r_1(1) + p_2 r_1(2) + p_3 r_1(3) = 0, \quad p_1 r_2(1) + p_2 r_2(2) + p_3 r_2(3) = 0 \tag{5}$$

Now let X be the outcome of the experiment, so X takes one of the three possible values $\{1, 2, 3\}$. If we now assign p_j to be the probability of outcome

j, then the expected value of the return from wager 1, assuming you bet one dollar, is

$$E[\mathbf{r}_1(X)] = p_1 r_1(1) + p_2 r_1(2) + p_3 r_1(3)$$

Hence (5) says that $E[\mathbf{r}_1(X)] = 0$, and also $E[\mathbf{r}_2(X)] = 0$. So the meaning of (5) is that there is some assignment of probabilities to the outcomes of the experiment which will make the expected returns for both wagers equal to zero.

To summarize: there are two possible situations. *Either* there is an arbitrage, meaning a betting scheme which is guaranteed to return a positive value for every outcome, *or else* there is an assignment of probabilities to the outcomes which makes the expected return zero for each wager.