Notes on Parrondo's game

Here I describe the three basic games for the case $M = 3$. Let W be the size of your current winnings. Every time you play, you toss a coin, and you Win if Heads comes up, while you Lose if Tails comes up. If you Win, then W increases by 1; if you Lose then W decreases by 1.

Game A This is the simplest case: you toss a biased coin which has probability p to come up Heads. So you Win with probability p, and you Lose with probability $1 - p$.

Game B The way you play depends on your current winnings. There are two cases:

a) if W is a multiple of 3, then you toss a coin with probability of Heads p_1.

b) if W is NOT a multiple of 3, then you toss a coin with probability of Heads p_2.

Game C At every play you randomly choose to play either Game A or Game B. You choose to play A with probability γ, and B with probability $1 - \gamma$.

The main work is to analyze Game B. Set it up as a 3-state Markov chain, where the states $\{0, 1, 2\}$ are the values of your current winnings modulo 3. Then at every play W changes by ± 1, and so you jump to a different state. The transition matrix is

$$P = \begin{pmatrix} 0 & p_1 & 1 - p_1 \\ 1 - p_2 & 0 & p_2 \\ p_2 & 1 - p_2 & 0 \end{pmatrix}$$

You want to find the probability of winning when you play Game B. Interpret this as the probability of winning after a long time, hence you want the probability of winning in the stationary distribution of P. You can find the stationary distribution of the chain, the probabilities are functions of p_1 and p_2. Then calculate the ratio of probability of winning over probability of losing, call this $P(W)/P(L)$. You want the condition on p_1, p_2 which guarantees that this ratio is bigger than one, which means that it is more likely that you win than lose. After some work with the algebra you can reduce this to the condition

$$\frac{P(W)}{P(L)} > 1 \quad \Leftrightarrow \quad \frac{(1 - p_1)(1 - p_2)^2}{p_1 p_2^2} > 1$$

After understanding Game B, you can now analyze Game C easily. Define

$$q_1 = \gamma p + (1 - \gamma) p_1, \quad q_2 = \gamma p + (1 - \gamma) p_2$$

and proceed in a similar way to Game B. You should get a similar condition for ratio of winning to losing probabilities. Now the idea is to adjust the values of p, p_1, p_2, γ so that both Games A and B are losing games, but Game C is winning. As a numerical example the following values should work:

$$p = 5/11; \quad p_1 = 1/121; \quad p_2 = 10/11; \quad \gamma = 1/2$$