MTH 7241: Fall 2009

Practice Problems for Midterm

1). Suppose that A, B and C are events in a sample space. Show that

$$P(A) + P(B) + P(C) \leq 2 + P(A \cap B \cap C)$$

2). A standard deck of 52 cards is shuffled and 5 cards are drawn at random. Let X be the number of aces remaining in the pack after this drawing. Find $E[X]$.

3). A fair die is rolled repeatedly. Let X_n be the result of the n^{th} roll. So X_n takes values $\{1, \ldots, 6\}$, each with probability $1/6$, and the random variables $X_1, X_2, \ldots$ are all independent.

Let

$$N = \min\{n : X_n = X_{n-1},\ n \geq 2\}$$

That is, N is the first roll where the result is equal to the previous roll. [e.g., if you roll the sequence $2, 3, 1, 4, 4, 6, \ldots$ then $N = 5$.]

Find $E[N]$.

4). A maze for rats is constructed with two doors; door 1 immediately leads to the exit, door 2 leads back to the maze after 1 minute. Assume that a rat is equally likely to choose either door at all times, and that if several rats are in the maze then they choose independently.
a). A rat is put in the maze. Find the expected time until he escapes.
b). Two rats are put in the maze. Find the expected time until the first escape occurs, and find the expected time until both escape.
c). Suppose n rats are put in the maze. Find the expected time until the first escape occurs.

[Hint: you may want to condition on the first choices made by all the rats].

5). A biased coin has probability p of coming up Heads. The coin is tossed n times, and the number of changeovers is counted. Call this number N_n. (Recall that a changeover occurs when Heads is followed by Tails, or vice versa). Show that $E[N_n] = (n-1)2p(1-p)$, and find the variance of N_n.

6). Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 0.5 & 0.3 & 0 & 0 & 0.2 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.4 & 0.4 & 0.2 & 0 \\ 0.3 & 0 & 0.2 & 0 & 0.5 \\ 0.5 & 0.2 & 0 & 0 & 0.3 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

a). Find and classify the equivalence classes of the states.

b). Find a stationary distribution for the chain. Is it unique?

c). Compute the expected number of steps needed to first return to state 1, conditioned on starting in state 1.

d). Compute the expected number of steps needed to first reach any of the states $\{1, 2, 5\}$, conditioned on starting in state 3.

7). In a certain game, a player who bets \$n either wins \$2n (with probability 1/2) or loses the \$n. So her total either increases by \$n or decreases by \$n, each with probability 1/2. A gambler uses the following strategy. She bets either \$10 (if her total is \$50 or more) or \$5 (if her total is less than \$50). If she starts with \$100, what is the probability that she reaches \$200 before losing everything?

8). Let $X_1, X_2, \ldots$ be defined jointly on some probability space, and suppose that $\mathrm{E}[X_i] = 0$ and $\mathrm{E}[(X_i)^2] = 1$ for all i. Prove that

$$P(X_n \geq n \text{ i.o.}) = 0$$

9). Let $X_1, X_2, \ldots$ be independent random variables and suppose that X_n is uniform on the set $\{1, 2, \ldots, n\}$ for each $n \geq 1$. Compute $P(X_n = 5 \text{ i.o.})$.