MTH G7241

Assignment 1: $\boxed{\text{SOLUTIONS}}$

Ex. #1: "only E is true" $= E \cap F^c \cap G^c$

"both E, F not G" $= E \cap F \cap G^c$

"at least two" $= (E \cap F) \cup (E \cap G) \cup (F \cap G)$.

Ex. #4: $P(E^c \cap F^c) = 0.6 \Rightarrow P(E \cup F) = 0.4$

$P(E \cap F) = 0.2$

$P(\text{exactly one is true}) = P(E \cup F) - \cancel{2}P(E \cap F)$

$= 1 - 0.6 - \cancel{2}(0.\cancel{2}) = 0.2$

Ex. #7: iterate $P(A \cap B) = P(A \mid B)\, P(B)$

Ex. #8: A_k = pile #k has exactly one Ace

want $P(A_4 \cap A_3 \cap A_2 \cap A_1)$

$$= P(A_4 \mid A_3 \cap A_2 \cap A_1)\; P(A_3 \mid A_2 \cap A_1)\; P(A_2 \mid A_1)\; P(A_1)$$

Now:

$$P(A_4 \mid A_3 \cap A_2 \cap A_1) = 1$$

$$P(A_3 \mid A_2 \cap A_1) = \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}}$$ ← # choices for 1 Ace and 12 others.

← total # choices

$$P(A_2 \mid A_1) = \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}}$$

$$P(A_1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}.$$

Ex#18: N = # tosses until first Head $\Rightarrow$ geometric $P(N=n) = (1-p)^{n-1}p$

X = # Heads in N tosses

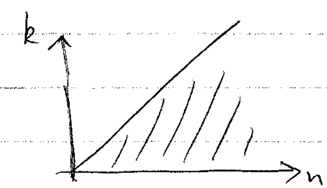
$$P(X=k) = \sum_{n=1}^{\infty} P(X=k \mid N=n)\, P(N=n)$$

$$= \sum_{n=k}^{\infty} P(X=k \mid N=n)\, P(N=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot p\, (1-p)^{n-1}$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^{k+1} (1-p)^{2n-k-1}.$$

Ex. #19: (in class)



Ex. #22:

$$E[N] = \sum_{n=1}^{\infty} n\, P(N=n)$$

$$= \sum_{n=1}^{\infty} P(N=n) \sum_{k=1}^{n}$$

$$= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P(N=n) = \sum_{k=1}^{\infty} P(N \geq k).$$

Ex # 23.

$$E[X^2] = \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} p = \frac{2-p}{p^2}.$$

Ex. # 32.

$$Y = \sum_{n=1}^{N} X_n$$

$$E[Y] = E[E[Y/N]]$$

$$= E[N\,E[X]] = \mu\,E[N]$$

$$E[Y^2] = E[E[Y^2/N]]$$

Now $E[Y^2/N=n] = E\left[\left(\sum_{k=1}^{n} X_k\right)^2\right]$

$$= \sum_{k=1}^{n}\sum_{m=1}^{n} E[X_k X_m]$$

$$= \sum_{k=1}^{n} E[X_k^2] + \sum_{k\neq m} E[X_k]\,E[X_m]$$

$$= n\,E[X^2] + (n^2-n)\mu^2.$$

$$\Rightarrow E[Y^2] = E[N\,E[X^2] + (N^2-N)\mu^2]$$

$$= E[N]\,E[X^2] + \mu^2 E[N^2] - E[N]\,\mu^2.$$

$$\Rightarrow \quad VAR[Y] = E[Y^2] - E[Y]^2$$

$$= E[N]\,E[X^2] + E[N^2]\,\mu^2 - E[N]\,\mu^2 - E[N]^2\,\mu^2$$

$$= E[N]\,VAR[X] + VAR[N].\,\mu^2.$$