THE BORSUK-ULAM THEOREM AND APPLICATIONS PRESENTED BY ALEX SUCIU AND MARCUS FRIES

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1. History

The Borsuk-Ulam theorem is one of the most applied theorems in topology. It was conjectured by Ulam at the Scottish Café in Lvov. Applications range from combinatorics to differential equations and even economics. The theorem proven in one form by Borsuk in 1933 has many equivalent formulations. One of these was first proven by Lyusternik and Shnirel'man in 1930.

2. Borsuk-Ulam

Theorem 2.1. *For $n > 0$ the following are equivalent:*

(i) *For every continuous mapping $f : \mathbb{S}^n \to \mathbb{R}^n$ there exists a point $\mathbf{x} \in \mathbb{S}^n$ such that $f(\mathbf{x}) = f(-\mathbf{x})$.*
(ii) *For every antipode-preserving map $f : \mathbb{S}^n \to \mathbb{R}^n$ there is a point $\mathbf{x} \in \mathbb{S}^n$ satisfying $f(\mathbf{x}) = \mathbf{0}$.*
(iii) *There is no antipode-preserving map $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$.*
(iv) *There is no continuous mapping $f : \mathbf{B}^n \to \mathbb{S}^{n-1}$ that is antipode-preserving on the boundary.*
(v) *Let $A_1, \ldots, A_d$ be a covering of $\mathbb{S}^d$ by closed sets A_i. Then there exists i such that $A_i \cap (-A_i) \neq \emptyset$.*

Proof.
($i \Rightarrow ii$) Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be an antipode-preserving map. By (i) there is a point $\mathbf{x} \in \mathbb{S}^n$ such that $f(\mathbf{x}) = f(-\mathbf{x})$. Since f is antipode-preserving we know $f(-\mathbf{x}) = -f(\mathbf{x}) = f(\mathbf{x})$, thus $2f(\mathbf{x}) = 0$ and $f(\mathbf{x}) = 0$.
($ii \Rightarrow i$) Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be a continuous map. Define a map $g : \mathbb{S}^n \to \mathbb{R}^n$ by $g(\mathbf{x}) = f(x) - f(-\mathbf{x})$. We see that $g(-\mathbf{x}) = -g(\mathbf{x})$, hence g is antipode preserving. By (ii) there is a point $\mathbf{x} \in \mathbb{S}^n$ such that $g(\mathbf{x}) = 0$ and thus $f(\mathbf{x}) - f(-\mathbf{x}) = 0$.

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($ii \Rightarrow iii$) Let $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ be an antipode-preserving map. We may compose f with the inclusion $i\colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$. By ($ii$) there is $\mathbf{x} \in \mathbb{S}^n$ such that $f(\mathbf{x}) = 0$. This is a contradiction since we assumed that $f(\mathbb{S}^n) \subset \mathbb{S}^{n-1}$.
($iii \Rightarrow ii$) Let $f\colon \mathbb{S}^n \to \mathbb{R}^n$ be an antipode-preserving map. Assume that $f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathbb{S}^n$. We may then define a map $g\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ by $g(\mathbf{x}) = \frac{f(\mathbf{x})}{\|f(\mathbf{x})\|}$. We see that g is an antipode-preserving map from $\mathbb{S}^n \to \mathbb{S}^{n-1}$, which contradicts ($iii$).
($iv \Rightarrow iii$) The map $\pi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$ is a homeomorphism from the upper hemisphere of $\mathbb{S}^n$ to $\mathbf{B}^n$. An antipode-preserving map $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ would yield a map $g\colon \mathbf{B}^n \to \mathbb{S}^{n-1}$ by $g(\mathbf{x}) = f(\pi^{-1}(\mathbf{x}))$ which is antipode-preserving on the boundary.
($iii \Rightarrow iv$) Assume $g\colon \mathbf{B}^n \to \mathbb{S}^{n-1}$ is antipode-preserving on the boundary. Then we can define a map $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ by $f(\mathbf{x}) = g(\pi(\mathbf{x}))$ for $\mathbf{x}$ in the upper hemisphere and $f(-\mathbf{x}) = -g(\pi(\mathbf{x}))$. We see that f is antipode-preserving which is a contradiction.
($i \Rightarrow v$) For a closed cover $F_1, \ldots, F_{n+1}$ of $\mathbb{S}^n$ we define a function $f\colon \mathbb{S}^n \to \mathbb{R}^n$ by $f(\mathbf{x}) = (\mathrm{dist}(\mathbf{x}, F_1), \ldots, \mathrm{dist}(\mathbf{x}, F_n))$. By ($i$) there is a point $\mathbf{x} \in \mathbb{S}^n$ such that $f(\mathbf{x}) = f(-\mathbf{x}) = \mathbf{y}$. If the i^{th} coordinate of $f(\mathbf{x})$ is non-zero then $\mathbf{x} \in F_i$. If all coordinates are non-zero then $\mathbf{x} \in F_{n+1}$.
($v \Rightarrow iii$) We first note that there exists a covering of $\mathbb{S}^{n-1}$ by closed sets $F_1, \ldots, F_{n+1}$ such that $F_i \cap (-F_i) = \emptyset$ for all i. To find such a cover consider the $n-$simplex in $\mathbb{R}^n$ centered at 0. Then project the faces of the $n-$simplex to the sphere. With this result in hand we see that if a continuous antipode-preserving map $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ existed, the sets $f^{-1}(F_1), \ldots, f^{-1}(F_{n+1})$ would be a cover of $\mathbb{S}^n$ such that $f^{-1}(F_i) \cap (-f^{-1}(F_i)) = \emptyset$. This contradicts ($v$), thus no such map can exist. □

Even though we have shown the equivalence of the above statements we have not shown that one of them is true in its own right. We will do that in the following

Theorem 2.2. *There is no antipode-preserving map $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$.*

Proof. Let $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ be an antipode-preserving map. Since f commutes with the map $i_k\colon \mathbb{S}^k \to \mathbb{S}^k$ given by $i_k(\mathbf{x}) = -\mathbf{x}$ we may descend to the quotient space. The quotient of $\mathbb{S}^k$ under i is the space

$\mathbb{RP}^k$. Thus we obtain a commutative diagram of spaces

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{f} & \mathbb{S}^{n-1} \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p_{n-1}} \\ \mathbb{RP}^n & \xrightarrow{\overline{f}} & \mathbb{RP}^{n-1} \end{array}$$

where $\overline{f}$ is the map induced on the quotients. So we find that an antipode-preserving map from $\mathbb{S}^n \to \mathbb{S}^{n-1}$ gives rise to a map $\overline{f}\colon \mathbb{RP}^n \to \mathbb{RP}^{n-1}$.

The existence of an antipode-preserving map f gives rise to a map in cohomology

$$\overline{f}^*\colon H^*(\mathbb{RP}^{n-1}; \mathbb{Z}_2) \to H^*(\mathbb{RP}^n; \mathbb{Z}_2).$$

To make use of this map we need to recall the following

$$H^*(\mathbb{RP}^k; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{k+1}), \qquad \deg(x) = 1,$$

where $\mathbb{Z}_2$ are the integers mod 2. Which when combined with the result above results in a ring homomorphism

$$\overline{f}^*\colon \mathbb{Z}_2[x]/(x^n) \to \mathbb{Z}_2[y]/(y^{n+1}).$$

We now claim that

$$\overline{f}^*(x) = y,$$

this will be proven later. Assuming the claim we find that

$$0 = \overline{f}^*(x^n) = y^n \neq 0,$$

which is a contradiction.

To complete the proof we need to show that $\overline{f}^*(x) = y$. As a first approximation we want to find $\overline{f}_*\colon \pi_1(\mathbb{RP}^n) \to \pi_1(\mathbb{RP}^{n-1})$. Using the path lifting criterion for the covering $p_n\colon \mathbb{S}^n \to \mathbb{RP}^n$ we can show that this is the identity map. Applying the Poincaré-Hurewicz theorem we find $\overline{f}_*\colon H_1(\mathbb{RP}^n; \mathbb{Z}) \to H_1(\mathbb{RP}^{n-1}; \mathbb{Z})$ is the identity map. Finally by the Universal Coefficient theorem for cohomology we find $\overline{f}^*\colon H^1(\mathbb{RP}^{n-1}; \mathbb{Z}_2) \to H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ is the identity map. □

3. The $\mathbb{Z}_2$ Index

Definition 3.1. A $\mathbb{Z}_2$-*space* is a pair (X, ν) with X a topological space and ν a homeomorphism $\nu\colon X \to X$ such that $\nu \circ \nu = id_X$. We say the $\mathbb{Z}_2$-action is *free* if ν has no fixed points.

Definition 3.2. A $\mathbb{Z}_2$-*equivariant map* is a function f from a $\mathbb{Z}_2$-space (X, ν) to a $\mathbb{Z}_2$-space (Y, μ) such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \nu \downarrow & & \downarrow \mu \\ X & \xrightarrow{f} & Y \end{array}$$

For brevity by $\mathbb{Z}_2$-map we mean a $\mathbb{Z}_2$-equivariant map.

Example 3.3. Let $(\mathbb{S}^n, \alpha_n)$ be a $\mathbb{Z}_2$-space where $\alpha_n(\mathbf{x}) = -\mathbf{x}$ is the antipode map. Then the Borsuk-Ulam theorem says that there is no $\mathbb{Z}_2$-equivariant map $f\colon (\mathbb{S}^n, \alpha_n) \to (\mathbb{S}^m, \alpha_m)$ if $m < n$. When we have $m \geq n$ there do exist $\mathbb{Z}_2$-equivariant maps given by inclusion.

The existence or non-existence of a $\mathbb{Z}_2$-map allows us to define a quasi-ordering on $\mathbb{Z}_2$-spaces motivated by the following

Definition 3.4. Let (X, ν) and (Y, μ) be $\mathbb{Z}_2$-spaces. If there exists a $\mathbb{Z}_2$-map $f\colon X \to Y$ we write

$$X \leq_{\mathbb{Z}_2} Y.$$

Simply stating that this is a quasi-ordering is not enough we need to check the properties.

Lemma 3.5. *The relation $\leq_{\mathbb{Z}_2}$ defined above is a quasi-ordering. That is, the ordering is reflexive and transitive.*

Proof. To see that $\leq_{\mathbb{Z}_2}$ is reflexive we use the identity map $\mathrm{id}_X\colon X \to X$. We see that this map commutes with any $\mathbb{Z}_2$-action. For transitivity let $f\colon X \to Y$ and $g\colon Y \to Z$ be $\mathbb{Z}_2$-maps. We then have the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \nu \downarrow & & \mu \downarrow & & \eta \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

$\square$

Using this quasi-ordering we are now in a position to define two numerical invariants associated to a $\mathbb{Z}_2$-space.

Definition 3.6. We define the $\mathbb{Z}_2$-*index* of a $\mathbb{Z}_2$-space (X, ν) by

$$\mathrm{ind}_{\mathbb{Z}_2}(X, \nu) = \min\{n \mid X \leq_{\mathbb{Z}_2} \mathbb{S}^n\}.$$

Dual to the index is the $\mathbb{Z}_2$-*coindex* defined by

$$\mathrm{coind}_{\mathbb{Z}_2}(X, \nu) = \max\{n \mid \mathbb{S}^n \leq_{\mathbb{Z}_2} X\}.$$

Where in both cases the $\mathbb{Z}_2$-action on $\mathbb{S}^n$ is given by the antipode map.

Proposition 3.7. *The $\mathbb{Z}_2$-index and coindex satisfy the following properties*

(i) $(X,\nu) \leq_{\mathbb{Z}_2} (Y,\mu) \Rightarrow \mathrm{ind}_{\mathbb{Z}_2}(X,\nu) \leq \mathrm{ind}_{\mathbb{Z}_2}(Y,\mu)$
(ii) $(X,\nu) \leq_{\mathbb{Z}_2} (Y,\mu) \Rightarrow \mathrm{coind}_{\mathbb{Z}_2}(X,\nu) \leq \mathrm{coind}_{\mathbb{Z}_2}(Y,\mu)$,
(iii) $\mathrm{coind}_{\mathbb{Z}_2}(\mathbb{S}^n,\alpha_n) = \mathrm{ind}_{\mathbb{Z}_2}(\mathbb{S}^n,\alpha_n) = n$,
(iv) *for all $\mathbb{Z}_2$-spaces (X,ν) we have* $\mathrm{coind}_{\mathbb{Z}_2}(X,\nu) \leq \mathrm{ind}_{\mathbb{Z}_2}(X,\nu)$.

Proof.
$(i), (ii)$ Assume $(X,\nu) \leq_{\mathbb{Z}_2} (Y,\mu)$. This means there is a $\mathbb{Z}_2$-map $f\colon X \to Y$. Let $g\colon Y \to \mathbb{S}^n$ be a $\mathbb{Z}_2$-map. If we consider the composition we obtain a map $g \circ f\colon X \to \mathbb{S}^n$. Thus we see that $\mathrm{ind}_{\mathbb{Z}_2}(X,\nu) \leq \mathrm{ind}_{\mathbb{Z}_2}(Y,\mu)$. Similarly for the coindex.
(iii) By Borsuk-Ulam we know that if $f\colon (\mathbb{S}^n,\alpha_n) \to (\mathbb{S}^m,\alpha_m)$ then we must have $n \leq m$. Combining this with the definitions of the index and coindex we obtain our result.
(iv) Assume that $\mathrm{coind}_{\mathbb{Z}_2}(X,\nu) = n$ and $\mathrm{ind}_{\mathbb{Z}_2}(X,\nu) = m$. We then have a composition of $\mathbb{Z}_2$-maps

$$\mathbb{S}^n \to X \to \mathbb{S}^m,$$

and (iii) implies that $n \leq m$. □

From the proof we see that (iii) is a reformulation of the Borsuk-Ulam theorem.

We cannot always expect $\mathrm{coind}_{\mathbb{Z}_2}(X) = \mathrm{ind}_{\mathbb{Z}_2}(X)$ and in general this is not true. In the cases when they are equal though we have the following

Lemma 3.8. *If* $\mathrm{coind}_{\mathbb{Z}_2}(X,\nu) = \mathrm{ind}_{\mathbb{Z}_2}(X,\nu)$ *then*

$$\pi_n(X) \twoheadrightarrow \mathbb{Z}\ .$$

Example 3.9. By a theorem of Stolz [4] we know

$$\mathrm{ind}_{\mathbb{Z}_2}(\mathbb{RP}^3,\iota) = 2,$$

where the $\mathbb{Z}_2$-action ι is induced from multiplication by i on $\mathbb{S}^3 \subset \mathbb{C}^2$. Now since $\pi_2(\mathbb{RP}^3) = 0$ we know that $\mathrm{coind}_{\mathbb{Z}_2}(\mathbb{RP}^3,\iota) < 2$.

Exercise 1. Show $\mathrm{coind}_{\mathbb{Z}_2}(\mathbb{RP}^3,\iota) = 1$.

4. Ham Sandwiches

Of the many theorems that follow from the Borsuk-Ulam theorem, the Ham Sandwich theorem has some of the best applications to combinatorics. Like the Borsuk-Ulam theorem the Ham Sandwich theorem has many different formulations, though not all are equivalent.

Definition 4.1. A *finite Borel measure* μ on $\mathbb{R}^d$ is a measure such that all open subsets of $\mathbb{R}^d$ are measurable and $0 < \mu(\mathbb{R}^d) < \infty$.

Theorem 4.2 (Ham Sandwich for measures)**.** *Let $\mu_1, \ldots, \mu_d$ be finite Borel measures on $\mathbb{R}^d$ such that every affine hyperplane has measure zero. Then there exists a hyperplane h such that*

$$\mu_i(h^+) = \frac{1}{2}\mu_i(\mathbb{R}^d) \quad \forall i = 1, \ldots, d,$$

where h^+ is one of the half spaces defined by h.

Proof. Let $\mathbf{u} = (u_0, \ldots, u_d) \in \mathbb{S}^d$. If at least one of $u_1, \ldots, u_d$ is not zero, we assign to $\mathbf{u}$ the half space

$$h^+(\mathbf{u}) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid u_1x_1 + \cdots + u_dx_d \le u_0\}.$$

Further we set $h^+((1, 0, \ldots, 0)) = \mathbb{R}^d$ and $h^+((-1, 0, \ldots, 0)) = \emptyset$. We see that h^+ assigns opposite half spaces to antipodal points. We define a function $f \colon \mathbb{S}^d \to \mathbb{R}^d$ by setting the i^{th} coordinate to be

$$f_i(\mathbf{u}) = \mu_i(h^+(\mathbf{u})).$$

If we assume for the moment that f is a continuous function, the Borsuk-Ulam theorem tells us there is a point $\mathbf{u}_0$ such that $f(\mathbf{u}_0) = f(-\mathbf{u}_0)$. Thus $\partial h^+(\mathbf{u}_0)$ is our desired hyperplane.

To complete the proof we need to show that f is a continuous function. Let $(\mathbf{u}_n)_{n=1}^{\infty}$ be a sequence of points of $\mathbb{S}^d$ converging to $\mathbf{u}$. To show that f is continuous it is enough to show that it is continuous along the projections $\pi_i \colon \mathbb{R}^d \to \mathbb{R}$. So we want to show

$$\lim_{n\to\infty} \mu_i(h^+(\mathbf{u}_i)) = \mu_i(h^+(\mathbf{u})).$$

Let $\mathbf{x}$ be a point not on the boundary of $h^+(\mathbf{u})$. Then for n sufficiently large we know $\mathbf{x} \in h^+(\mathbf{u}_n)$ if and only if $\mathbf{x} \in h^+(\mathbf{u})$. So if g denotes the characteristic function of $h^+(\mathbf{u})$ and g_n the characteristic function of $h^+(\mathbf{u}_n)$, we have $g_n(\mathbf{x}) \to g(\mathbf{x})$ for all $\mathbf{x} \ni \partial h^+(\mathbf{u})$. Since $\partial h^+(\mathbf{u}_n)$ have measure zero we know that the g_n converge to g almost everywhere. By Lebesgue's dominated convergence theorem, we thus have $\mu_i(h^+(\mathbf{u}_n)) = \int g_n d\mu_i \to \int g d\mu_i = \mu_i(h^+(\mathbf{u}))$. □

For our applications we will need a discrete version of the Ham sandwich theorem. We present that now.

Theorem 4.3. *Let $A_1, \ldots, A_d \subset \mathbb{R}^d$ be finite point sets. Then there exists a hyperplane h that simultaneously bisects $A_1, \ldots, A_d$.*

By "h simultaneously bisects A_i we mean that each open half space defined by h contains at most $\lfloor \frac{1}{2}|A_i| \rfloor$ points.

Proof. We will prove the theorem by considering three cases, each more general than the previous. To begin we assume that the points are in general position in $\mathbb{R}^d$ and each A_i contains an odd number of points. At each point we center an ε-ball. We then choose $\varepsilon > 0$ so that no hyperplane intersects more that d of the ε-balls from one set A_i. Now by the Ham sandwich theorem we know there is a hyperplane which simultaneously bisects each set. Additionally since there are an odd number of points the hyperplane must intersect exactly one ball from each set, and this cut contains the center.

We now assume that each A_i contains an odd number of points but the points need not be in general position. For every $\eta > 0$ we define new sets $A_{i,\eta}$ by moving the points of A_i by at most η so that the points of $A_{1,\eta} \cup \cdots \cup A_{d,\eta}$ are in general position. By the previous argument there exists a hyperplane h_η which simultaneously bisects each $A_{i,\eta}$. We write $h_\eta = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{a}_\eta \rangle = b_\eta\}$ where $\mathbf{a}_\eta$ is a unit vector. The points b_η lie in a bounded interval thus by compactness there is a cluster point $(\mathbf{a}, b) \in \mathbb{R}^{d+1}$ of the pairs $(\mathbf{a}_\eta, b_\eta)$. Let h be the hyperplane defined by $(\mathbf{a}, b)$. Let $\eta_1 > \eta_2 > \cdots$ be a sequence of points converging to η. Now if $\mathbf{x}$ lies at distance 2δ from h then for i sufficiently large $\mathbf{x}$ lies distance δ from h_{η_i}. Thus if the open half space defined by h contains k points, there is j such that $i > j$ implies that the open half space defined by h_{η_i} contains k points.

Finally if some sets A_j contain an even number of points we simply delete one. We then apply the above argument to the remaining points. Then we return the point, noticing that this does not change the result due to the definition of bisection. □

Corollary 4.4 (Ham sandwich for general position sets)**.** *Let $A_1, \ldots, A_d$ be finite point sets in $\mathbb{R}^d$ such that $A_1 \cup \ldots \cup A_d$ is in general position. Then there exists a hyperplane h such that each open half space contains exactly $\lfloor \frac{1}{2}|A_i| \rfloor$ points.*

Proof. We begin with an arbitrary ham sandwich hyperplane h as given by the ham sandwich theorem. The problem is that h may contain up to d points of some A_i.

Fix a coordinate system so that h is given by $x_d = 0$. Let $B = h \cap (A_1 \cup \ldots \cup A_d)$. We know that B consists of at most d affinely independent points. We want to move h slightly so that it is our desired cut. Since the points of B are affinely independent we may make them stay on h, go above or go below.

To see this add $d - |B|$ new points to B so that we obtain a d-point affinely independent set $C \subset h$. For each $\mathbf{a} \in C$ we choose a point $\mathbf{a}'$ such that either $\mathbf{a}' = \mathbf{a}$ or $\mathbf{a}' = \mathbf{a} + \varepsilon \mathbf{e}_d$ or $\mathbf{a}' = \mathbf{a} - \varepsilon \mathbf{e}_d$. We let

$h' = h(\varepsilon)$ be the hyperplane defined by the points $\mathbf{a}'$. Now for all $\varepsilon > 0$ sufficiently small, the $\mathbf{a}'$ remain affinely independent and the motion of $h(\varepsilon)$ is continuous in ε. Thus we can guarantee that for all sufficiently small $\varepsilon > 0$, h' is our desired hyperplane. □

5. Lunch

All this discussion about ham sandwiches has built up our appetite. Now is the time to feast.

Theorem 5.1 (Akiyama and Alon 1989)**.** *Consider sets $A_1, \dots, A_d$ of n points each, in general position in $\mathbb{R}^d$. Let $\{1, \dots, d\}$ be a set of colors and color the points of A_i with color i. Then the points of the union $A_1 \cup \dots \cup A_d$ can be partitioned into "rainbow" d-tuples with pairwise disjoint convex hulls.*

Proof. We proceed by induction on the size of the sets A_i. If $n = 1$ we take the convex hull of the set $A_1 \cup \cdots \cup A_d$. Now assume that $n > 1$ and odd. Then by (4.4) there exists a hyperplane which intersects each A_i in exactly one point and each open half space contains at most $\lfloor \frac{1}{2}|A_i| \rfloor$ points. Let the first d-tuple be the points on the hyperplane. Then apply the inductive hypothesis to each open half space. If $n > 1$ and even (4.4) guarantees there is a hyperplane which bisects each A_i but does not intersect any A_i. We then apply the inductive hypothesis to the resulting open half spaces. □

Theorem 5.2 (Necklace theorem)**.** *Every open necklace with d kinds of stones, an even number of each, can be divided among two thieves using no more than d cuts.*

Before proving the theorem we need a preliminary result.

Lemma 5.3. *Any affine hyperplane in $\mathbb{R}^d$ cuts the curve $\gamma(t) = (t, t^2, \dots, t^d)$ in at most d points.*

The curve $\gamma(t)$ is commonly referred to as the *moment curve*. We leave the proof of the lemma as an exercise to the reader. We now give a proof of the theorem.

Proof. We place the necklace into $\mathbb{R}^d$ along the moment curve $\gamma(t)$. We then define sets A_i by

$$A_i = \{\gamma(k) \mid \text{ the } k\text{-th stone is of the } i\text{-th kind}\}.$$

By the general position ham sandwich theorem there is a hyperplane h which bisects each A_i. Additionally by the lemma we know h cuts $\gamma(t)$ in at most d points. □

Definition 5.4. Let N be a set of n points, for each $1 \leq k \leq n$ write $N_k :=\{S \mid S \subset N, |S| = k\}$. Define the *Kneser graph* $KG_{n,k}$ to be the graph with vertex set N_k and edges given by $uv \in E(KG_{n,k})$ if and only if $u \cap v = \emptyset$.

Theorem 5.5 (Lovász 1978)**.** *For all $k > 0$ and $n > 2k - 1$, the chromatic number of the Kneser graph is given by*

$$\chi(KG_{n,k}) = n - 2k + 2.$$

Proof.
Upper bound: For a vertex $v \in N_k$ color v by

$$\phi(v) = \min\{\min(v), n - 2k + 2\}.$$

This defines a proper coloring since if $\phi(v) = \phi(u) = i < n-2k+2$ then $i \in v \cap u$ and hence uv is not an edge. If we have $\phi(v) = \phi(u) = n-2k+2$ then u, v are subsets of the set $\{n-2k+2, \dots, n\}$, but this set contains $2k - 1$ elements thus $u \cap v \neq \emptyset$.

Lower bound: Set $d = n-2k+1$. Let $X \subset \mathbb{S}^d$ be a set of n points such that no hyperplane passing through the origin contains more than d points of X. Let the vertex set of $KG_{n,k}$ be identified with the set X_k.

Assume there is a proper coloring of $KG_{n,k}$ with at most $n - 2k + 1 = d$ colors. Fix one such coloring and define sets $A_1, \dots, A_d \subset \mathbb{S}^d$ by $\mathbf{x} \in A_i$ if there is at least one k-tuple of color i inside the open hemisphere $H(\mathbf{x})$ centered at $\mathbf{x}$. Finally define $A_{d+1} = \mathbb{S}^d \setminus (A_1 \cup \dots \cup A_d)$. Clearly the sets $A_1, \dots, A_d$ are open and A_{d+1} is closed. By the general version of the Lyusternik-Shnirel'man theorem there is a set A_i such that $\mathbf{x}, -\mathbf{x} \in A_i$. If $i \leq d$ we obtain two disjoint k-tuples with the same color, thus the coloring is not proper. If $i = d + 1$ we know $H(\mathbf{x})$ contains at most $k - 1$ points of X and $H(-\mathbf{x})$ contains at most $k - 1$ points of X. Therefore the complement $\mathbb{S}^d \setminus (H(\mathbf{x}) \cup H(-\mathbf{x}))$ contains at least $n - 2k + 2 = d + 1$ points. But the complement is defined by a hyperplane passing through the origin, which contradicts our assumption that no hyperplane through the origin contains more that d points. □

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