Higher derivatives of forms and level Artinian algebras

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1 Motivating Question

An Artinian algebra A will be an associative algebra with identity 1 having finite dimension as a vector space over a given field K: we will term this dimension the length of A.

$V \subset S_j$ where V is a vector space, $dimV = d$. and $S = k(x_1, ..., x_r)$. $D(V)$ is the space of first partials of $v \in V$. The interest lies in the various partial derivations of forms in V.

Can $dim_k D(V) < dim_k(V) = d$

1. $d \leq deg(V)$
2. $d > deg(V)$

2 What is an Artinian Algebra?

2.1 Artinian algebras as thick points

Example 2.1 $B = k[x, y]/(x^2, xy, y^2)$. Evaluate $f \in B$,$f = a + bx + cy$. We know $f(p)$, $f_x(p)$, $f_y(p)$, $p = (0, 0)$. First order neighborhood around p.

Example 2.2 $A = k[x, y]/(x, y^3)$ A function f can be represented by $f = a + by + cy^2 modI$ where $I = (x, y^3)$. f is determined by $f(p)$, $f_y(p)$, $f_{y^2}(p)$.

So it may be viewed as a thick point, the origin elongated with a higher order part in the y direction.

Example 2.3 $A_\lambda = k[x,y]/(x+\lambda y^2, y^3)$ for $f \in A_\lambda$, know $f(p)$, $f_y(p)$ and $\lambda f_x(p) - f_{y^2}(p)$.

2.2 Singularity Theory

Let f be a mapping germ $f : (C^r, 0) \to (C^s, 0)$, with $f = (f_1, \ldots, f_s)$, let $Q_f = k[x1, \ldots, x_r]/(f_1, \ldots, f_s)$.

Theorem 2.4 *$dim_k Q_f$ finite => Q_f gives the right-left equivalence class of f.*

2.3 Cohomology rings of projective manifolds M

Theorem 2.5 *$H^*(M)$=Gorenstein Artinian algebras, $H^*(M) = k[x_1, \ldots, x_r]/I$ with variables having weights not all one.*

Example 2.6 (V. Puppe) There exists a 6 dimension mainfold M with no orientation preserving symmetries (or S^1 action).
M. Kreck (2004): N has no orientation reversing involution. The proof is based on "generic singularity" Gorenstein algebras A, $H(A) = (1, 6, 6, 1)$.

3 The family $LevAlg(H) \subset GradAlg(H)$

Let $A = R/I$, $R = k[x_1, \ldots, x_r]$. Let $M = (x_1, \ldots, x_r)$. $I_i = I \bigcap M^i / I \bigcap M^{i+1}$, $A_i = R_i/I_i$.

Definition 3.1 I graded $\iff$ $I = \bigoplus I_i$.
$H(A) = (h_0, h_1, \ldots, h_i, \ldots, h_j)$, $h_i = dim_k A_i$.

The socle of an Artinian algebra is the subspace

$$Soc(A) = (0 : M)_A = \{a \in A | M_a = 0\}$$

type $t(A) = dim_k Soc(A)$
A Artinian: socle $deg(A) = max\{j|A_j\} \neq 0\}$

Example 3.2 $A = k[x,y]/(xy, x^3 - y^3)$ is graded, level, $H(A) = (1,2,2,1)$,
$Soc(A) = A_3 =< \overline{x^3} >$
$A \cong < 1; x, y; x^2, y^2; x^3 >$ as a vector space.
$t(A) = 1 \iff A$ is Gorenstein.

Example 3.3 $A = k[x,y,z]/(xy, x^2 - y^2, z^3, M^4)$, $H(A) = (1,3,4,3)$,
A level, type 3.

3.1 Macaulay duality

Goal: Construct level algebras!

Let $R = k[x,y,z]$ act on $S = k[X,Y,Z]$ as PDO's:

$$f \circ g = f(\partial/\partial X, \partial/\partial Y, \partial/\partial Z)(g)$$

.

Let $W \subset S_j$ be a vector space $I_W = Ann(W) = \{f | f \circ w = 0, \forall w \in W\}$

Lemma 3.4 (Macaulay, or classical apolarity) *$A = R/I$ level, type t, socle deg $j \iff I = I_W$, $W \subset S_j$, $dimW = t$.*

Here $W = I_j^{\perp} = \{w \in S_j | I_j \circ w = 0\}$.
$\widehat{A} = Hom_k(A,k) \cong R \circ W$ is the dual module to A. $H(A)_i = dim_k(A_i) = dim_k R_{j-i} \circ W$

Example 3.5 $S = k[X,Y,Z]$, $W =< ZX^2 + ZY^2, Z^2X, Z^2Y >$,

$$I_W = (xy, x^2 - y^2, z^3, M^4)$$

$$\widehat{A} =< 1; X, Y, Z; X^2 + Y^2, ZX, ZY, Z^2; W >.$$

$$H(A) = (1,3,4,3)$$

4 Which H occur for level algebras?

4.1 case $t = 1$, Gorenstein sequences

$W =< F >$
Fact: $H(A)_i = H(A)_{j-i}$

Q1: Can H be non-unimodal ($t = 1$)?

Hint: need $r \geq 4$

Answer: (R. Stanley): yes: $r = 13, H = (1, 13, 12, 13, 1)$.
Let $R = k[u_1, \dots, u_{10}, x, y, z]$,
$S = k[U_1 \dots U_{10}, X, Y, Z]$
$F = \Sigma_{i=1}^{10} U_i \mu_i = U_1 X^3 + \cdots + U_{10} Z^3$
$\mu = (X^3, X^2Y, X^2Z, XY^2, \dots Z^3)$.
$R_2 \circ F \subset (x^2, xy, \dots, z^2) \circ F + < X^2, XY, \dots, Z^2 >$.
So $H_2 \leq 12$.
Answer: (D. Bernstein) $r \geq 5$. yes.

$H = (1, 5, \dots, 91, 90, 91, \dots, 1)$. This is analogous to the Stanley construction: use level algebra B and dual $\hat{B}$.

Problem: Unimodality, t= 1, Open for r=4.

Answer: H is unimodal, $r = 3$. (R. Stanley using Buchsbaum, Eisenbud, Pfaffian Structure Theorem.) $\Delta H_i = H_i - H_{i-1}$

Theorem 4.1 *$r = 3$, $t = 1$, socle degree j. H is a Gorenstein Sequence $\iff T = \Delta H_{\leq j/2}$ is an O-sequence height 2.*

$T = (1, 3, \dots, \nu, t_\nu, t_{\nu+1}, \dots t_{j'})$ with $\nu \geq t_{\nu+1} \geq \dots \geq t_{j'}$.

Example 4.2 *$H = (1, 3, 6, 9, 12, 14, 15, 17, 17, 16, 15, 12, 9, 6, 3, 1)$, $j = 15$. Here $T = (1, 2, 3, 3, 3, 3, 2, 2, 1)$.*

Theorem 4.3 (Buchsbaum-Eisenbud structure) *$t = 1$, $r = 3$.
$I = (\sqrt{M_{1,1}}, \sqrt{M_{2,2}}, \dots, \sqrt{M_{m,m}})$ $M_{i,i}$ = the diagonal minor of alternating mxm M, $m = 2v + 1$*

Definition 4.4 *PGor(H)$\subset P(S_j)$ projective space: parametrizes Gorenstein quotients A or R, H(A)=H.*

Theorem 4.5 (S. Diesel, 1996) *$r = 3$, PGor(H) is irreducible.*

4.2 Which H occur, for level algebras, $t > 1$?

Theorem 4.6 *$r = 2$, $H(A) = (1, 2, \ldots, \nu, h_\nu, \ldots, h_j = t)$,*
Let $e_i = H_{i-1} - h_i$. A level implies

$$0 \leq e_\nu \leq \ldots \leq e_j \qquad (*)$$

Any H satisfying (*) occurs for some A.

Example 4.7 $H = (1, 2, 3, 4, 5, 6, 7, 6, 4, 2)$ *possible for level.* $H = (1, 2, 3, 4, 5, 3, 2)$ *not possible.*

5 Mostly Accessible Problems

1. What are Gorenstein sequences H, $r = 4$, $t = 1$?
2. Cohomology ting of projective manifolds GrAlg(H), $r = 2$. Same for LevAlg(H), $r = 2$
3. Closure of postulation strata $Z(H) \subset Hilb^H(P^2)$
4. Classification: Can KevAKbg(H), $r = 3$, $t = 2$ have several irreducible components?
5. Tighten connection iwth singularities
6. Nongraded level algebras, Gorenstein algebras: what are H, $r = 3$?
7. Quantum topologies and Gorenstein algebras
8. Hilbert scheme of points:
   (a) on P^2 or surfaces
   (b) on P^r
9. Hilber schemes and ideal of extremal growth.
10. Closure of strata Z(H) (all ideals), in $Hilb^n k\{x, y\}$.
11. Local $Hilb^n k\{x, y\}/J$