SINGULARITIES
Presented by David Massey; May 4, 2005.

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Question: How are we to describe topological spaces?

- To begin:
  - We understand $\mathbb{R}^n$.
  - We understand manifolds because they are locally Euclidean.
  - We understand submanifolds of $\mathbb{R}^n$.
  - Example: A "cusp" in $\mathbb{R}^2$, given by $y^2 = x^3$, is "topologically" a submanifold of $\mathbb{R}^2$. However, it's not smooth.
  - We've examined these submanifolds since ninth grade, under the name conic sections.
- Definition: Given f, some polynomial function of $\underline{x} := (x_1, \ldots x_n)$, an Algebraic Hypersurface is $V(f) = f^{-1}(0)$, if the dimension is one less than the ambient dimension.
  - The Implicit Function Theorem tells us that, given $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}$, and $p \in U$; if p is not a critical point of f, then $f^{-1}(f(p))$ is a smooth $(n-1)$ dimensional submanifold of U near p. (Note that a critical point is one where $\frac{\partial f}{\partial \underline{x}}(p) = 0$.)
  - It is when f has critical points that one gets interesting local topology. For example, $f = xy$, where $V(f)$ is the coordinate axes, and there exists one critical point, the origin. (The origin is a singularity of $V(f)$.) In this example, our hypersurface is not smooth.

We have a problem in that the real numbers don't behave well. For example, take $f = x^2 + y^2$, where $V(f)$ has dimension 0, as opposed to 1, as one would desire or expect. A possibly more troubling example occurs for $g = y^2 - zx^2$, because $V(g)$ has a 1-dimensional piece and a 2-dimensional piece; i.e. it's not pure-dimensional.

Consequently, let's move to complex analytic functions, with U an open subset of $\mathbb{C}^{n+1}$, and $f : (U, \underline{0}) \to (\mathbb{C}, \underline{0})$ such that $f \not\equiv 0$. In this setting, $V(f)$ is purely n-dimensional, i.e., there exists an open, dense subset of $V(f)$ which is a (complex analytic) n-submanifold of U. We'll refer to those points which are not singularities of $V(f)$ as **smooth points** of $V(f)$.

- Example 1: Let $f = xy$, and consider $V(f)$ in $\mathbb{C}^2$. Take a small ball around the origin, $B_\epsilon = \{\underline{z} \mid |\underline{z}| \leq \epsilon\}$ and a small sphere, $S_\epsilon = \partial B_\epsilon = \{\underline{z} \mid |\underline{z}| = \epsilon\}$.
- Theorem: $(B_\epsilon,\, B_\epsilon \cap V(f),\, p) \cong \text{cone}(S_\epsilon,\, S_\epsilon \cap V(f))$. (The cone point is the third element on the right side.)

So, the embedding of $S_\epsilon \cap V(f)$ into S_ϵ completely describes the local ambient topology.

- Definition: $K = S_\epsilon \cap V(f)$ is the **real link** of $V(f)$ at $\underline{0}$.
  - Return to Example 1. Then, $S_\epsilon = S^3$ and $K = S^1 \cup S^1 \hookrightarrow S^3$. These two circles are linked in S^3; this is called the Hopf Link.
  - Example 2: $f = y^2 - x^3$. In this case, K is a trefoil knot in S^3.
  - Example 3: $V(2xy - z^2) \subseteq \mathbb{C}^3$. Using the substitution $x = s^2$, $y = t^2$, $z = \sqrt{2}st$, one can see that you get the same point (x, y, z) for (s, t) and $(-s, -t)$. Further, $|x|^2 + |y|^2 + |z|^2 = (|s| + |t|)^2$; thus $K \cong \mathbb{RP}^3$. As an exercise, show that this is not a topological manifold. $B_\epsilon \cap V(f) \cong \text{cone}(\mathbb{RP}^3)$.

To further understand this embedding of $S_\epsilon \cap V(f)$ in S_ϵ, it is extremely useful to examine it's compliment, $S_\epsilon \setminus K$.

- Theorem (Milnor, 1968): $\frac{f}{|f|} : S_\epsilon \setminus K \to S^1$ is a smooth, locally trivial fibration.

This means that over each $v \in S^1$, $\left(\frac{f}{|f|}\right)^{-1}(v)$ is diffeomorphic to a fixed space $F_{f,\underline{0}}$, and there exists a diffeomorphism $h : F_{f,\underline{0}} \to F_{f,\underline{0}}$ (called a characteristic diffeomorphism of the fibration) such that $\frac{f}{|f|} : S_\epsilon \setminus K \to S^1$ is diffeomorphic to

$$\frac{F_{f,\underline{0}} \times [0,1]}{(x,0) \sim (h(x),1)} \to \frac{[0,1]}{0 \sim 1}.$$

The map h induces an automorphism, h^*, on $H^*(F_{f,\underline{0}})$, called the monodromy automorphism. Although the homeomorphism h is not necessarily unique, the induced h^* *is* unique.

- Definition: $F_{f,\underline{0}}$ is the Milnor Fiber of f at $\underline{0}$, and the fibration is the Milnor Fibration.

The Milnor Fibration yields important information about $S_\epsilon \setminus K$; in particular, we have the homotopy long exact sequence of a fibration:

$$\ldots \to \pi_i(F_{f,\underline{0}}) \to \pi_i(S_\epsilon \setminus K) \to \pi_i(S^1) \to \pi_{i-1}(F_{f,\underline{0}}) \to \ldots$$

and the cohomological Wang Sequence:

$$\ldots \to H^i(S_\epsilon \setminus V(f)) \to H^i(F_{f,\underline{0}}) \xrightarrow{id - h^*} H^i(F_{f,\underline{0}}) \to H^{i+1}(S_\epsilon \setminus V(f)) \to \ldots$$

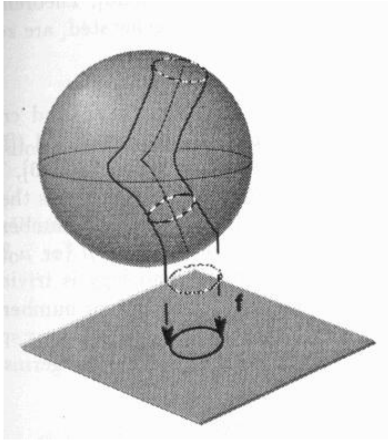
Let $\partial\mathbb{D}_\eta$ denote the boundary of the closed disc of radius η around the origin of $\mathbb{C}$. Then

$$\frac{f}{|f|} : S_\epsilon \setminus K \to S^1$$

is homeomorphic to

$$f : \overset{\circ}{B}_\epsilon \cap f^{-1}(\partial\mathbb{D}_\eta) \to \partial\mathbb{D}_\eta.$$

This last characterization gives us a "Milnor Tube," so to speak, artistically represented here.[1]



- Facts: $F_{f,\underline{0}}$ has the homotopy type of a finite n-dimensional CW complex. It is a real, $2n$-dimensional smooth manifold. In particular, $H_i\left(F_{f,\underline{0}}\right) = 0$ if $i > n$, and $H_n\left(F_{f,\underline{0}}\right)$ is free Abelian.

Let $s := \dim_{\underline{0}} \Sigma f$, the dimension of the critical locus of f at $\underline{0}$. Then $F_{f,\underline{0}}$ is $(n-s-1)$-connected. Now, for $s = 0$, $F_{f,\underline{0}}$ is $(n-1)$-connected and so $F_{f,\underline{0}} \simeq_h \bigvee_{\mu} S^n$ where μ is defined to be the Milnor number of f at $\underline{0}$.

- Theorem:

$$\mu_0\left(f\right) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{z\}}{\left\langle \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n} \right\rangle}$$

  where $\mathbb{C}\{z\}$ is the ring of convergent power series at $\underline{0}$.

- Example: For $f = y^2 - x^3$, $\mu_0\left(f\right) = 2$ because $\frac{\mathbb{C}\{x,y\}}{\langle -3x^2, 2y\rangle} \cong \frac{\mathbb{C}\{x\}}{\langle x^2\rangle}$, which has $[1]$ and $[x]$ as a basis. On the other hand, for $f = y^2 - x^3 - x^2$, $\mu_0\left(f\right) = 1$. In a family, the constancy of μ implies constant topological type.

Wrap up: Suppose $s = 1$, a situation called a "non-isolated singularity." It could have non-trivial $H^{n-1}\left(F_{f,\underline{p}}\right)$ and $H^n\left(F_{f,\underline{p}}\right)$, for all $\underline{p} \in \Sigma f$. The question to ask is: Does data at a nearby point $\underline{p}$ have any implication for $H^*\left(F_{f,\underline{0}}\right)$? The answer is yes, and the tool we have at our disposal is a sheaf, a device for encoding how local data patches together to give global data. Here, we need a constructible complex of sheaves of $\mathbb{Z}$-modules, to see how nearby Milnor Fibers affect Milnor Fibers at the origin.

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[1] "The Milnor Fibration Inside a Ball" from page 1 of D. Massey, *Lê Cycles and Hypersurface Singularities*, Springer-Verlag 1995.