On Variety: The Grassmannian, Schubert, and Determinental Varieties*

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1 Algebraic Varieties

Let k be an algebraically closed field. $\mathbb{A}^n$ denotes the affine n-space, which is the set $\{(a_1, \ldots a_n) \,|a_i \in k\}$. Let I be an ideal in $k[x_1, \ldots x_n]$.

Definition 1.1 *The set* $V(I) = \{(a_1, \ldots a_n) \in \mathbb{A}^n | f(a_1, \ldots a_n) = 0 \;\forall\; f \in I\}$ *is an* affine variety.

Conversely, let $X \subset \mathbb{A}^n$, define $\mathcal{I}(X) = \{f \in k[x_1, \ldots x_n] \,|f(x) = 0 \;\forall\; x \in X\}$.

Let $\mathbb{P}^n = \{\mathbb{A}^{n+1} \setminus \{0\}\}/\sim$, where $(a_0, \ldots a_n) \sim (\lambda a_0, \ldots \lambda a_n)$ for any $\lambda \in k \setminus \{0\}$. This is the Projective n-space. Now let I be a homogeneous ideal in $k[x_0, \ldots x_n]$, meaning I is generated by homogeneous polynomials. In general, it does not make sense to evaluate a polynomial f at a point p of the Projective n-space. However, if f is homogeneous of degree d, then $f(\lambda a_0, \ldots \lambda a_n) = \lambda^d f(a_0, \ldots a_n)$, and since $\lambda \neq 0$, we can talk about f being zero or non-zero at a point p.

Definition 1.2 *The set* $V(I) = \{(p) \in \mathbb{P}^n | f(p) = 0$ *for all homogeneous* $f \in I\}$ *is a* projective variety.

Conversely, for $X \subset \mathbb{P}^n$, let $\mathcal{I}(X)$ be the ideal generated by $\{f \in k[x_0, \ldots x_n], f$ homogeneous $|f(p) = 0 \;\forall\; p \in X\}$.

Definition 1.3 *For* X *an affine variety in* $\mathbb{A}^n$, *the* coordinate ring *of* X *is* $k[x_1, \ldots x_n]/\mathcal{I}(X)$, *also denoted* $k[X]$.

Coordinate rings of Projective varieties have a similar definition.

Definition 1.4 *For* X *a variety,* $f \in k[X]$, *the* principal open subsets *are given by* $X_f = \{x \in X | f(x) \neq 0\}$.

Definition 1.5 *The* Zariski Topology *on* $\mathbb{A}^n$ *or* $\mathbb{P}^n$ *has the principal open subsets as a base.*

*Presentation based on Lakshmibai and Gonciulea's *Flag Varieties*, Hermann, Èditeurs des Sciences et des Arts, 2001.

2 The Grassmannian Variety and its Schubert Varieties

Fix integers $1 \leq d < n$, let $V = k^n$, with basis $\{e_1, \ldots e_n\}$.

Definition 2.1 *The* Grassmannian $G_{d,n}$ *is the set of d-dimensional subspaces* $U \subset V$*.*

If $U \in G_{d,n}$ with $a_1, \ldots a_d$ as a basis, then U may be represented by the $n \times d$ matrix $A = (a_{ij})$, of rank d, whose columns are vectors $a_1, \ldots a_d$. This matrix is not unique, as it depends on choice of basis for U.

Definition 2.2 *Let* $I_{d,n} = \{\underline{i} = (i_1, \ldots i_d) \,|1 \leq i_1 < \ldots < i_d \leq n\}$.

Define a partial order $\leq$ on $I_{d,n}$ by defining $\underline{i} \leq \underline{j} \Leftrightarrow i_t \leq j_t \,\forall\, t$. Let $N = \binom{n}{d}$, the order of $I_{d,n}$.

The exterior product map $\Lambda^d : V \oplus \ldots \oplus V \longrightarrow V \wedge \ldots \wedge V$ such that $(a_1, \ldots a_d) \mapsto a_1 \wedge \ldots \wedge a_d$ induces an embedding

$$f_d : G_{d,n} \longrightarrow \mathbb{P}\left(\Lambda^d V\right) = \mathbb{P}^{N-1}$$

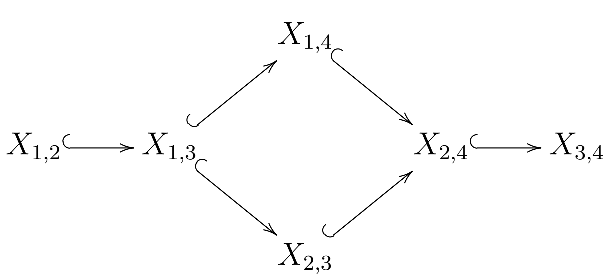
called the Plücker map. We will index the coordinates of $\mathbb{P}^{N-1}$ by the set $I_{d,n}$. Thus, for $p \in \mathbb{P}^{N-1}$, $p_{\underline{i}}$ will denote the $\underline{i}^{th}$ coordinate of p, called the Plücker coordinates. For $U \in G_{d,n}$, represented by matrix A as above, $p_{\underline{i}}(U) = \det(A_{\underline{i}})$, where $A_{\underline{i}}$ represents the $d \times d$ sub-matrix of A with rows $i_1, \ldots i_d$. One can see that this embedding is well-defined, as different representative matrices for the same point will be mapped to the same projective coordinates. Furthermore, the Plücker embedding is injective, so we will often identify $G_{d,n}$ with its embedding in $\mathbb{P}^{N-1}$.

The Plücker embedding of $G_{d,n}$ consists of the zeroes of the Plücker quadratic relations. For example, the Plücker quadratic relation for $G_{2,4}$ is $p_{1,4}p_{2,3} = p_{2,4}p_{1,3} - p_{3,4}p_{1,2}$. (There is a general form for the Plücker relations for any Grassmannian which we will leave out here.) We can now assert that the Grassmannian is in fact a Projective variety.

Definition 2.3 *For* $1 \leq t \leq n$*, let* $V_t \subseteq V$ *spanned by* $\{e_1, \ldots e_t\}$*. For every* $\underline{i} \in I_{d,n}$*, define the Schubert Variety associated to* $\underline{i}$ *as* $X_{\underline{i}} = \{U \in G_{d,n} | \dim(U \cap V_{i_t}) \leq t,\ 1 \leq t \leq d\}$.

Remark 1 $X_{\underline{i}} \subseteq X_{\underline{j}} \Leftrightarrow \underline{i} \leq \underline{j}$.

Example: $G_{2,4}$



Remark 2 ***Bruhat Decomposition:***

$$X_{\underline{j}} = \overline{\bigcup_{\underline{i} \leq \underline{j}} B \cdot e_{\underline{i}}},$$

where $B \subseteq GL(n, k)$ *is the set of all upper triangular matrices, and they are acting on the point* $e_{\underline{i}} \in G_{d,n}$ *with basis* $e_{i_1}, \ldots e_{i_d}$*. The closure refers to the Zariski Topology.*

Remark 3 $p_{\underline{j}}|_{X_{\underline{i}}} \neq 0 \Leftrightarrow \underline{i} \geq \underline{j}$.

The above remark assures us that the Schubert variety does in fact fit the definition of a projective variety. It's defining ideal is generated by the Plücker quadratic relations as well as the monomials implied above.

Remark 4 $\dim X_{\underline{i}} = \sum i_t - t$.

At this point in our study of Schubert varieties, it would be useful to find an affine "cell," or "patch" of the Schubert variety. Bruhat Decomposition gives us a starting point from which to find this cell, since matrix groups can often be easily associated with affine spaces. The key, however, is to find a set that intersects with every Schubert variety.

Definition 2.4 *The* Opposite Cell *in* $G_{d,n}$*, denoted* $\mathcal{O}^-$*, is the orbit* $B^- \cdot [e_1 \wedge \ldots \wedge e_d]$ *where* B^- *is the group of lower triangular matrices in* $GL(n,k)$*.*

With some thought, one can see that $\mathcal{O}^-$ is easily identified with $\left[\begin{smallmatrix} I_{d\times d} \\ A_{r\times d}\end{smallmatrix}\right]$, I being the identity matrix, A being any matrix, and $r = n - d$. Thus, $\mathcal{O}^-$ includes the single point of the minimal Schubert variety, (represented by $\left[\begin{smallmatrix} I_{d\times d} \\ 0_{r\times d}\end{smallmatrix}\right]$), and therefore intersects every other Schubert variety at at least this point. Also, $\mathcal{O}^-$ is equivalent to $\mathbb{A}^{rd}$, relating to our goal of finding an affine patch.

Definition 2.5 *The Opposite Cell in a Schubert Variety* $X_{\underline{w}}$ *is* $\mathcal{O}^- \cap X_{\underline{w}}$*, denoted* $Y_{\underline{w}}$*.*

It's helpful to note that $Y_{\underline{w}} = (X_{\underline{w}})_{(p_{1,\ldots d})}$, which can be seen by the matrix representation of $\mathcal{O}^-$. This helps us to see how $Y_{\underline{w}}$ is affine.

3 The Determinantal Variety, D_t

Let $X = (x_{ij})$ be an $r \times d$ matrix of indeterminants. Let $I_t(X)$ be the ideal in $k[x_{ij}]$ generated by all possible t-minors of X, $t \leq \min(r,d)$.

Definition 3.1 *Let* D_t *be a closed subvariety of* $\mathbb{A}^{mn} (\cong M_{m,n})$ *with* $I_t(X)$ *as the defining ideal.*

The determinantal variety has been known and studied classically. It is a relatively recent development, however, to relate it to the Schubert variety. Before stating this relation, we need to show a bijection between the minors of $X_{r,d}$ and the Plücker coordinates of $G_{d,n}$. Take $\underline{i} \in I_{d,n}$, let m be such that $i_m \leq d$, and $i_{m+1} > d$. Let

$$\begin{aligned} A_{\underline{i}} &= \{n+1-i_d,\ n+1-i_{d-1}, \ldots\ n+1-i_{m+1}\} \\ B_{\underline{i}} &= \text{ The complement of } \{i_1, \ldots i_m\} \text{ in } \{1, \ldots d\} \end{aligned}$$

Now, $\theta : I_{d,n} \setminus \{(1, \ldots d)\} \to$ all minors of X by setting $\theta(\underline{i})$ = the minor of X with rows given by $A_{\underline{i}}$ and columns given by $B_{\underline{i}}$. (Conventionally, one associates the smallest element of $I_{d,n}$, $(1, \ldots d)$ with the constant 1, or the "empty-minor" of X.) Since the Plücker

coordinates are numerated by $I_{d,n}$, we have a bijection between the Plücker coordinates and minors of X.
Example, $G_{3,6}$:

$$\mathcal{O}^- \quad = \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}$$

$$p_{(1,2,6)} \quad \overset{\theta}{\longleftrightarrow} \quad x_{1,3} \text{ (viewed as a 1-minor)}$$

$$p_{(2,4,6)} \quad \overset{\theta}{\longleftrightarrow} \quad x_{1,1}x_{3,3} - x_{1,3}x_{3,1}$$

Assuming $t \leq \min\{r, d\}$, let $\underline{w} = (t, t+1, \ldots d, n+2-t, n+3-t, \ldots n) \in I_{d,n}$.

Theorem 1 $D_t \cong Y_{\underline{w}} = X_{\underline{w}} \cap \mathcal{O}^- = (X_{\underline{w}})_{(p_{1,\ldots d})}$.

The proof, though not given here, consists of showing that the defining ideals are actually the same for D_t and $Y_{\underline{w}}$. The bijection θ is used here. This theorem is useful in that properties known about one variety are now known about the other. We will use it to find the dimension of the determinantal variety. Since $Y_{\underline{w}}$ is open in $X_{\underline{w}}$, it has the same dimension as the Schubert variety.

$$\begin{aligned} \dim D_t = \dim Y_{\underline{w}} &= \sum w_t - t \\ &= t-1+(t+1)-2+\ldots+d-(d-t+1)+n+2-t-(d-t+2)+\ldots \\ &\quad +n-d \\ &= (t-1)(d-t+1)+(n-d)(n-(n+2-t+1)) \\ &= (t-1)(d-t+1)+(n-d)(t-1) \\ &= (t-1)(r+d-t+1) \qquad \text{(because } n=r+d) \end{aligned}$$

4 The Singular Locus

Definition 4.1 *For X an affine variety and $R = k[X]$, $\mathcal{O}_{X,x}$ is the* stalk *at $x \in X$, if $\mathcal{O}_{X,x} = R_{\mathcal{P}}$, where $\mathcal{P}$ is the prime ideal $\{f \in R | \, f(x) = 0\}$.*

Hence $\mathcal{O}_{X,x}$ is a local ring.

Definition 4.2 *The point $x \in X$ is* non-singular *if $\mathcal{O}_{X,x}$ is a regular local ring.*

Definition 4.3 *The singular locus, denoted Sing X, refers to all singular points of X, (those points which are not non-singular).*

Now, assume $\mathcal{I}(X)$ is the ideal defining X in $\mathbb{A}^n$, and is generated by $\{f_1, \ldots f_r\}$. Let $\mathcal{J}$ be the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$.

The *Jacobian criterion* says $\operatorname{rank}\mathcal{J}_p \leq \operatorname{co}\dim_{\mathbb{A}^n} X$ with equality if and only if p is a non-singular point of X. By $\mathcal{J}_p$ we mean the matrix $\mathcal{J}$ evaluated at the point p.

Theorem 2 *Sing* $D_t = D_{t-1}$

Proof: Let $\mathcal{J}$ be the Jacobian Matrix of D_t. Let $\mathcal{M}_t$ denote the set of all t-minors of the generic $r \times d$ matrix X. Since $\mathcal{I}(D_t)$ is generated by the elements of $\mathcal{M}_t$, we'll index the rows of $\mathcal{J}$ by $\mathcal{M}_t$. and thus the columns can be indexed by the set of all variables in X. (So $\mathcal{J}$ has rd columns, and $\binom{r}{t} \cdot \binom{d}{t}$ rows.) Let $M \in \mathcal{M}_t$ and τ be a variable in X. Then the $(M, \tau)^{\text{th}}$ entry in $\mathcal{J}$ is non-zero if and only if τ appears in M, in which case the $(M, \tau)^{\text{th}}$ entry is equal to $\pm M'$, where M' is the $(t-1)$-minor obtained from M by deleting the row and column containing τ.

Let z be an $r \times d$ matrix, (i.e. an element of $\mathbb{A}^{rd}$).

Case 1 Assume $z \in D_{t-1}$, implying that all $(t-1)$-minors of X vanish at z. Then $\mathcal{J}_z = 0$, since entries of $\mathcal{J}$ are either 0 or $(t-1)$-minors of X, and by the Jacobian criterion, $z \in$ Sing D_t. Therefore, $D_{t-1} \subseteq$ Sing D_t.

Case 2 Now assume $z \notin D_{t-1}$, implying that there exists a $(t-1)$-minor M of X that is non-zero at z. M contains $(t-1)$ rows and $(t-1)$ columns of X. Let $\mathcal{B}$ denote the set of all variables of X not apearing in any row or column given by M. Let $\tau \in \mathcal{B}$, and let M_τ be the t-minor of X obtained by adding the row and column of τ to M. Now, (M_τ, τ) is an entry in $\mathcal{J}_z$, and is equal to $\pm \det M(z)$, which we've assumed to be non-zero. For any $\sigma \in \mathcal{B}$, $\sigma \neq \tau$, (M_τ, σ) in $\mathcal{J}_z$ is zero.
Thus, the minor in $\mathcal{J}_z$ with rows given by $\{M_\tau, \tau \in \mathcal{B}\}$ and columns given by $\mathcal{B}$ is non-zero, (it could be rearranged to form a diagonal matrix). Thus, rank$\mathcal{J}_z \geq \#\mathcal{B} = (r-t+1)(d-t+1) = rd - (t-1)(r+d-(t-1)) = \operatorname{co}\dim_{\mathbb{A}^{rd}} D_t$. We know that rank$\mathcal{J}_z$ is always less than or equal to $\operatorname{co}\dim_{\mathbb{A}^{rd}} D_t$, so we must have equality. By the Jacobian criterion, z is a non-singular point of D_t.

$\Rightarrow$ Sing $D_t = D_{t-1}$.