TILING SPACE
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Abstract. The study of plane tilings can be traced to the ancient Greek mathematicians. Generalizations of this problem both to higher dimensions and general polytopes has continued to the modern day. The purpose of this article is to give an introduction to the theory of tilings. We begin by covering the definitions and terminology of the subject. We then discuss some of the major results and problems.

1. Terminology and Definitions

Definition 1.1. A *tiling* is a countable family of closed subsets $\mathcal{T}$ of $\mathbb{E}^d$ which cover $\mathbb{E}^d$ without gaps or overlaps.

Throughout, the tiles are assumed to be topological d-balls. We will also assume that our tiling is locally finite. In other words, for any point a ball centered at that point intersects only finitely many tiles.

There are many examples of tilings. One of the most common is tiling $\mathbb{E}^d$ by "cubes" of equal size. Another is given by tiling $\mathbb{E}^2$ by regular hexagons, this occurs in nature as a honeycomb. Of course these tilings are very special in that they look the same at every vertex. This symmetry can be recorded in the form of a group.

Definition 1.2. Let $\mathcal{T}$ be a tiling of $\mathbb{E}^d$. Define the *symmetry group of* $\mathcal{T}$ by

$$S(\mathcal{T})\colon = \{\phi \mid \phi \text{ is an isometry of } \mathbb{E}^d \text{ such that } \phi(\mathcal{T}) = \mathcal{T}\}.$$

Definition 1.3. Let $\mathcal{T}$ be a tiling of $\mathbb{E}^d$.

(i) **normal:** We say that $\mathcal{T}$ is normal if the tiles are uniformly bounded in size.
(ii) **periodic:** $\mathcal{T}$ is said to be periodic if $S(\mathcal{T})$ contains translations in d independent directions.
(iii) **sub-periodic:** $\mathcal{T}$ is said to be sub-periodic if $S(\mathcal{T})$ contains between 1 and $d-1$ inclusive, independent translations.

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(iv) **non-periodic:** $\mathcal{T}$ is said to be non-periodic if $S(\mathcal{T})$ contains no translations other than the identity.

(v) **regular:** $\mathcal{T}$ is said to be regular if $S(\mathcal{T})$ acts transitively on the flags of $\mathcal{T}$.

(vi) **face-to-face:** A tiling $\mathcal{T}$ by convex polytopes is said to be face-to-face if any two tiles intersect in a face of each (possibly the empty face).

(vii) **isohedral:** A tiling $\mathcal{T}$ is said to be isohedral if $S(\mathcal{T})$ is transitive on the tiles of $\mathcal{T}$.

Tiling the plane by hexagons is a normal, periodic and regular tiling. In fact in $\mathbb{E}^2$ the only regular tilings are tiling by triangles, squares or hexagons.

Minkowski made the observation that if the tiles of $\mathcal{T}$ are convex, then each tile is a convex polyhedron. If additionally the tiles are compact then the tiles are polytopes.

Definition 1.4. Let $\mathcal{T}$ be a tiling. A *prototile set of* $\mathcal{T}$ is a minimal subset of tiles of $\mathcal{T}$ such that each tile of $\mathcal{T}$ is congruent to one of the tiles in the set. A tiling is *monohedral* if there is a single prototile.

Question 1 (Space Filler Problem)**.** Classify all convex polytopes that admit a monohedral tiling. Open for dimensions greater than or equal to 2.

1.1. **Periodic Tilings.** Recall that a periodic tiling is a tiling of $\mathbb{E}^d$ such that the symmetry group has translations in d independent directions.

Definition 1.5. A discrete subgroup G of the group of isometries of $\mathbb{E}^d$ is called *crystallographic* if $\mathbb{E}^d/G$ is compact.

The symmetry group of a periodic tiling is a crystallographic group, but two or more periodic tilings may have the same symmetry group. To construct two periodic tilings with the same crystallographic group we need only dissect the fundamental region of the group in different ways. Though we must take care because different dissections may lead to the same tiling!

We note, since $\mathbb{E}^d/G$ is compact and our tiling is assumed to be locally finite there can be only finitely many prototiles.

The natural questions at this point are: How do we identify crystallographic groups and how many are there in each dimension? The following theorem gives a partial answer.

Theorem 1.6 (Bieberbach)**.**

(i) *G is crystallographic if and only if the subgroup Λ of translations in G is of finite index and has rank d.*
(ii) *For every d there are only finitely many different types of crystallographic groups (up to conjugacy in the group of non-singular affine transformations).*
(iii) *Any two crystallographic groups in $\mathbb{E}^d$ are of the same type if and only if they are abstractly isomorphic as groups.*

The number of crystallographic groups in a given dimension is an open problem for dimensions greater than 4. Below we list the number of crystallographic groups for dimensions 1-4:

d	# types
1	2
2	17
3	219
4	4783
≥ 5	?

The 17 crystallographic groups in dimension 2 are sometimes called the *wallpaper groups*. These 17 groups are the symmetry groups for wallpaper.

1.2. **Classification by Symmetry.** Recall that a tiling is regular if $S(\mathcal{T})$ is transitive on the flags of $\mathcal{T}$. The following table lists the regular tilings of $\mathbb{E}^d$.

d	Tiles
1	unit intervals
2	triangles, squares, hexagons
4	generalized octahedra, 24-cells, 4-cubes
d≥ 1	cubes

The property of being a regular tiling is very restrictive. We may expand our class of tilings by loosening the restriction of flag transitivity. A *uniform plane tiling* is a tiling of $\mathbb{E}^2$ by regular polygons which is edge-to-edge and $S(\mathcal{T})$ is vertex transitive. It can be shown that the uniform plane tilings are exactly the Archimedian plane tilings.

If we instead loosen the restrictions to consider plane tilings with "regular vertex neighborhoods" which are edge-to-edge and $S(\mathcal{T})$ is tile transitive we obtain a class of tilings called the *Laves tilings*. A remarkable fact is that the Laves tilings are dual to the Archimedian tilings.

Definition 1.7. A shape T is said to tile by translations if T admits a tiling of $\mathbb{E}^d$ by translations. If the set of translation vectors forms a lattice , then $\mathcal{T}$ is called a *lattice tiling*.

The *Voronoi region of a lattice* Λ is the set

$$V(\Lambda, x) = \{y \in \mathbb{E}^d \mid |y - x| \leq |y - z| \forall z \in \Lambda\}$$

where $x \in \Lambda$. Then $\mathcal{T}$ defined by

$$\mathcal{T} := \{V(\Lambda, x) | x \in \Lambda\}$$

is a tiling of $\mathbb{E}^d$. Additionally Λ acts transitively by translations.

Theorem 1.8. *If a convex d-polytope T tiles $\mathbb{E}^d$ by translations, then T admits a unique face-to-face lattice tiling.*

Note that the above theorem does not hold for T not convex.

A *parallelotope* is a convex polytope that tiles by translations.

1.3. **Tilings by Cubes (translates).** In 1907 Minkowsi asked whether in a lattice tiling $\mathcal{T}$ by unit cubes, does there always exist a "stack" of cubes in which each two adjacent cubes meet in a face? This was answered in the positive by Hajos in 1947. Following Minkowski, in 1930 Keller asked whether in a tiling $\mathcal{T}$ by unit cubes, does there always exist a pair of tiles which meet in a facet? A counterexample was found in dimension 10 by J. Lagarias and P. Shor. Finally we may ask, is there a polyhedral tile which admits a monohedral tiling but no isohedral tiling? The answer to this is yes.

2. Fundamental Question

Question 2. Are there only finitely many (combinatorial) types of space-fillers in $\mathbb{E}^d$?

The answer to the above question is unknown in general. In dimension 1 the answer is yes, and the only space filler in this case is the unit interval. In dimension two we know that all triangles, quadrangles and certain pentagons and hexagons are space fillers, but the list of pentagons may not be complete. For dimensions greater than two the problem is wide open. In dimension three there is an example with 38 facets.

The following theorem gives one bound on the number of facets that a space filler may possess.

Theorem 2.1 (Delone, Tarasov)**.** *If a space filler admits an isohedral face-to-face tiling then*

$$\#\textit{facets} \leq 2^d(b - \frac{1}{2}) - 2,$$

where b is the index of the translation subgroup in $S(\mathcal{T})$.

In dimension three we have $d = 3$ and $b = 48$ which give us a bound of 378, quite a bit larger than the known example with 38 facets.

3. Tilings and Aperiodicity

Let $\mathcal{P}$ be a finite set of prototiles in $\mathbb{E}^d$.

Definition 3.1. A prototile set $\mathcal{P}$ is called *aperiodic* if all tilings of $\mathbb{E}^d$ by $\mathcal{P}$ are non-periodic.

It may be the case that there are tiles patches which cannot be extended by the tiles of $\mathcal{P}$. The natural question then is if we can tile arbitrarily large patches with the tiles of $\mathcal{P}$ can we tile all of $\mathbb{E}^d$. This may seem like a simple question since we are assuming that we may tile arbitrarily large patches but our given tiling may not be extendable beyond our given patch.

Theorem 3.2. *If $\mathcal{P}$ can tile over arbitrarily large $d-$balls, then $\mathcal{P}$ admits a tiling of the entire space $\mathbb{E}^d$.*

3.1. **Tiling Problem.**

Question 3. Does there exist an algorithm which, when applied to any finite set $\mathcal{P}$ of prototiles, decides in finitely many steps whether or not $\mathcal{P}$ admits a tiling of $\mathbb{E}^d$?

Conjecture 1 (Wang, 1960)**.** The tiling problem is decidable, i.e. there are no aperiodic prototile sets.

This conjecture was shown to be false. In 1966 Berger found a set of 20,426 tiles which form an aperiodic prototile set. The implication is that the tiling problem is undecidable!

Further examples of aperiodic prototile sets have been discovered by Robinson, Penrose and others. There is an example by Penrose of an aperiodic prototile set consisting of two prototiles.

Question 4. Does there exist an aperiodic monotile?

References

[1] B. Grünbaum, G. Shephard, *Tilings and Patterns* , W. H. Freeman and Company, New York, 1989. xii+446 pp. ISBN 0-7167-1998-3

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