

# Tilings on Cluster Algebras

Notes from a talk given by Gordana Todorov, typeset by Kristin Webster

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## 1 Introduction

$Q$  a finite quiver,  $K = \bar{K}$  algebraically closed.  $Rep_K Q$  is the category of finite dimensional representations of  $Q$  over  $K$ .

$Rep_K Q \simeq mod \Lambda$  is the category of finitely generated  $\Lambda$ -modules where  $\Lambda (= KQ)$  is a path algebra.

### 1.1 Gabriel's Theorem

1.  $Rep_K Q$  is of finite representation type if and only if the underlying quiver is of type  $A_n, D_n, E_6, E_7, E_8$
2. If  $Q$  is of type  $A_n, D_n, E_6, E_7, E_8$  then there is a 1-1 correspondence between positive roots of  $Q$  and the set of Isomorphic indecomposable  $\Lambda$ -modules.

### 1.2 Theorem (Dlab-Ringel)

1. Equivalent to Gabriels Theorem pt. 1,  $Q$  has finite representation type iff it is a dynkin diagram.
2. If  $Q$  is a Dynkin diagram of type  $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$  then there is a 1-1 correspondence between positive roots and isomorphism classes of indecomposable  $Q$ -representations.

Recall: The path algebra  $KQ = \Lambda$  is hereditary i.e.

- submodules of projective modules are projective
- $Ex^2(M, N) = 0$
- global dimension of  $\Lambda \leq 1$

## 2 Cluster Categories

- Do not depend on orientation
- Related to Cluster Algebras

Need:

- $D^b(\text{mod}\Lambda)$  derived category
- Auslander-Reiten translates  $\tau$
- Describe derived category  $D^b(\text{mod}\Lambda)$  Since  $\Lambda$  is hereditary it is easy to describe  $D^b$ .

### 2.1 Objects of $D^b$

$$\bigcup_{i \in \mathbb{Z}} (\text{mod}\Lambda)[i]$$

### 2.2 Morphisms

- $\text{Hom}_{D^b}(M, N) = \text{Hom}_{\Lambda}(M, N)$ ,  $M, N \subset \text{mod}\Lambda$
- $\text{Hom}_{D^b}(M, N[i]) = \text{Ext}_{\Lambda}^1(M, N)$
- $\text{Hom}_{D^b}(M, N[i]) = 0$ ,  $i \neq 0, 1$
- $\text{Hom}_{D^b}(M[j], N[i+j]) = \text{Hom}_{D^b}(M, N[i])$

There are 2 functors:  $[1] : D^b(\text{mod}\Lambda) \rightarrow D^b(\text{mod}\Lambda)$   $\tau^{-1}$  Define:  $F = [1]\tau^{-1} : \text{mod}\Lambda \rightarrow D^b(\text{mod}\Lambda)$   $X \rightarrow \tau^{-1}X[1]$

Cluster category objects are orbits of  $F$  on  $D^b$ . Morphisms:  $\text{Hom}_C(\tilde{X}, \tilde{Y}) = \bigoplus_i \text{Hom}_{D^b}(F^i X(\text{mod}\Lambda), Y)$

Representatives of orbits

Remark: Representations of indecomposable isomorphism classes of indecomposable objects in  $C$  may be chosen as indecomposable modules and first shifts of projectives.

### 2.3 Tilting Theory

- originally started with (Auslander, Platzeck and Reiten) APR tilts (which corresponds to certain simple reflections on roots.)
- Reflection functors

- Generalizations of tiltings
- Definition - Tilting module for hereditary algebra  $\Lambda$
- $Ext_{\lambda}(T, T) = 0$
- $T = \bigoplus_{i=1}^n T_i$ ,  $T_i$  indecomposable and non-isomorphic

Defintion: T in C is a tilting object if  $Ext_{\underline{C}}(T, T) = 0$   $T = \bigoplus_{i=1}^n T_i$ ,  $n$  is the number of vertices. Theorem(Hopper/Unger):  $\Lambda$  hereditary algebra,  $n$

= number of non-isomorphic simples If  $T = \bigoplus_{i=1}^n T_i$  and  $Ext(T, T) = 0$  then there is exactly one or two ways to complete it to a tilting module.

Theorem:  $\underline{C}$  is a cluster category of  $\Lambda$ . If  $T' = \bigoplus T_i$ , then there are exactly 2 ways to extend it to a tilting module.