Quiver Representations and Gabriel's Theorem

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1 Introduction

This talk is based on the 1973 article by Berstein, Gelfand and Ponomarev entitled *Coxeter Functors and Gabriel's Theorem* and also *Notes on Representations of Quivers* by Jerzy Weyman. A quiver is a graph with orientation where we interpret the vertices as vector spaces and the arrows as linear maps. Quivers have interesting relations to root systems.

Definition 1.1 *A quiver is a directed graph $Q = (Q_0, Q_1)$ where Q_0 is the set of vertices and Q_1 is the set of arrows. The arrows will be denoted by the letters of the alphabet. The arrow a has a head ha and a tail ta, which are in Q_0.*

$$ta \xrightarrow{a} ha$$

Example 1.2

$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \end{tikzcd}$$

$Q_0 = 1, 2, 3$, $Q_1 = a, b$, $ta = 1$, $ha = tb = 2$, $hb = 2$.

Example 1.3

$$\begin{tikzcd} 1 \arrow[r, shift left=2, "a"] \arrow[r, shift right=2, "b"'] & 2 \arrow[l, "c" description] \end{tikzcd}$$

Example 1.4

$$\begin{tikzcd} & 2 \arrow[d, "b"] \\ 1 \arrow[ur, "a"] & 3 \arrow[l, "c"] \end{tikzcd}$$

Definition 1.5 *K is an algebraically closed field. A finite dimensional representation V of Q is a set of $\{V(x) : x \in Q_0\}$ of finite dimensional K-vector spaces together with a set of K-linear maps $\{V(a) : V(ta) \to V(ha) | a \in Q_1\}$*

Definition 1.6 *A morphism $f : V \to V'$ of two representations is a collection of K-linear maps $f(x) : V(x) \to V'(x)$; $x \in Q_0$, such that for each $a \in Q_1$ the following diagram commutes:*

$$\begin{tikzcd} V(ta) \arrow[r, "V(a)"'] \arrow[d, "f(ta)"] & V(ha) \arrow[d, "f(ha)"] \\ V'(ta) \arrow[r, "V'(a)"'] & V'(ha) \end{tikzcd}$$

A morphism f is an isomorphism if f_x is invertible for every $x \in Q_0$

The representations of a quiver Q over K form a category denoted $Rep_K(Q)$.

Example 1.7 *Two representations of* $1 \xrightarrow{a} 2 \xrightarrow{b} 3$

$$\begin{tikzcd}
V & K \arrow[r, "id"] \arrow[d, "id"] & K \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[d, "0"] \\
V' & K \arrow[r, "0"] & 0 \arrow[r, "0"] & 0
\end{tikzcd}$$

Example 1.8 *Examples of a representation that is not isomorphic:*

$$\begin{tikzcd}
V' & K \arrow[r, "0"] \arrow[d, "?"'] & 0 \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[d, "0"] \\
V & K \arrow[r, "id"] & K \arrow[r, "0"] & 0
\end{tikzcd}$$

Example 1.9 *Consider*

$$ta \xrightarrow{a} ha$$

. Suppose that $V(ta) = K^m$ and $V(ha) = K^n$. If B and C are $m \times n$ matrices and $V_B(a) = B$ and $V_C(a) = C$, then V_C are isomorphic iff we have invertible maps $f(ta)$ and $f(ha)$ such that $f(ha)B = Cf(ta)$. This implies that $f(ha)Bf(ta)^{-1} = C$. So V_B and V_C are isomorphic quiver representations iff C can be obtained from B by a change of basis in K^m and K^n.

Definition 1.10 *The dimension vector of a representation V is the function d_V defined by $d_V := (dimV(x))$*

Example 1.11 *Use the above examples to illustrate dimension.*

and $V = K^3 \longrightarrow K^2 \longrightarrow K^5$ *has* $\underline{d}(V) = (3, 2, 5)$.

Definition 1.12 *If V and W are representations of a quiver Q, then we can define the direct sum representation $V \bigoplus W$ by$(V \bigoplus W)(x) = V(x) \bigoplus W(x)$ for every $x \in Q_0$ and$(V \bigoplus W)(a)$: $V(ta) \bigoplus W(ta) \to V(ha) \bigoplus W(ha)$ us described by the matrix* $\begin{pmatrix} V(a) & 0 \\ 0 & W(a) \end{pmatrix}$.

Definition 1.13 *A representation V is called decomposable if V is isomorphic to a direct sum of non-zero representations. Otherwise a representation is indecomposable.*

Since all the vector spaces are finite dimensional, every representation can be written as a finite sum of indecomposable representations.

Example 1.14 *Consider the quiver* $1 \longrightarrow 2 \longrightarrow 3$ *with* V_i *finite dimensional* K*-vector spaces. There are six indecomposable objects:*
Consider $sl(4)$ *with basis* $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ *on* A_3

$K \longrightarrow 0 \longrightarrow 0$, $\alpha_1 = \epsilon_1 - \epsilon_2$,
$0 \longrightarrow K \longrightarrow 0$, $\alpha_2 = \epsilon_2 - \epsilon_3$,
$0 \longrightarrow 0 \longrightarrow K$, $\alpha_3 = \epsilon_3 - \epsilon_4$,
$K \longrightarrow K \longrightarrow 0$, $\alpha_1 + \alpha_2 = \epsilon_1 - \epsilon_3$,
$0 \longrightarrow K \longrightarrow K$, $\alpha_2 + \alpha_3 = \epsilon_2 - \epsilon_4$,
$K \longrightarrow K \longrightarrow K$, $\alpha_1 + \alpha_2 + \alpha_3 = \epsilon_1 - \epsilon_4$,
With dimensions: ...
$K^3 \longrightarrow K^2 \longrightarrow K^5$ *is decomposable.*

Example 1.15 *Example:* D_4*, give an indecomposable representation.*

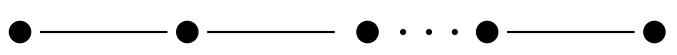
Example 1.16 *Example:* $\widetilde{D_4}$ *an example of infinitely many non-isomorphic representations (hence not a dynkin quiver).*

2 Gabriel's Theorem

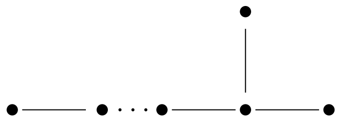
Definition 2.1 *A quiver with finitely many non-isomorphic indecomposable representations is said to be of finite representation type.*

Gabriel's Theorem: Gabriel in a 1972 article raised and solved the following problem, to find all graphs for which there exist only finitely many non-isomorphic indecomposable representations. He made the following observation: Part 1 of Gabriel's Theorem: A quiver is of finite representation type if and only if it is a Dynkin diagrams of type: (called Dynkin quivers)

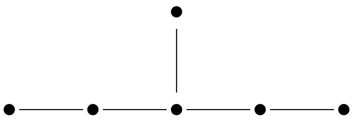
A_n



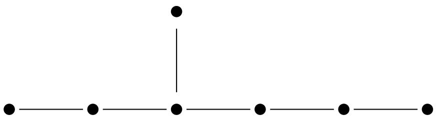
D_n



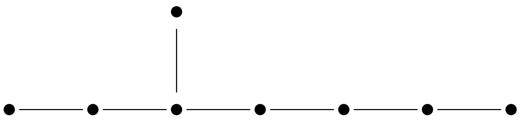
E_6



E_7



E_8



In particular the representation finite representation type property does not depend on the orientation of the arrows.

Part 2 of Gabriel's Theorem: The indecomposable representations correspond naturally to the positive roots of the corresponding root system.

The proof of this by Berstein, Gel'fand and Ponomarev in 1972 is based on the technique of roots and the Weyl group.

Consider the quadratic form: $B_Q(\alpha) =< \alpha, \alpha >= \sum_{x \in Q_0} \alpha(x)^2 - \sum_{a \in Q_1} \alpha(ta)\alpha(ha)$

$\Gamma(Q)$ is the graph obtained from Q by forgetting the orientation.

The first half of Gabriel's Theorem (if a quiver Q contains finitely many isomorphism classes then it is a Dynkin quiver), can be proved by the following 2 propositions:

Proposition 2.2 *Let Γ be a connected graph. The quadratic form B_Γ is positive definite if and only if Γ is a Dynkin graph of type A_n, D_n, E_6, E_7, E_8.*

Proof of the "only if" part: Begin by observing that if for some graph Γ, the form B_Γ is positive definite, then for all full subgraphs Γ', B_Γ' is also positive definite.

1. If B_Γ is positive definite then Γ does not contain loops.
   If Γ contains a loop, then it contains a full subgraph Γ' with one vertex and $m \geq 1$ loops at that vertex. If we take the dimension vector α whose value at the vertex is 1. Then $B_{\Gamma'}(\alpha) = 1 - m$, so it is not positive definite.

2. If B_Γ is positive definite and n is the number of vertices in Γ, and m a number of edges then $m < n$. That is 2 vertices can be connected by at most one edge. Consider the dimension vector $\alpha(x) = 1$ for $x \in \Gamma_0$. Then $B_\Gamma(\alpha) = n - m$.

3. Every vertex is connected to at most three vertices

   Let the vertex $x \in \Gamma_0$ be connected to vertices y_1, y_2, y_3, y_4. Consider the full subgraph Γ' with vertices x, y_1, y_2, y_3, y_4. This subgraph can contain only four edges connecting x to y_i. Take $\alpha(x) = 2, \alpha(y_i) = 1$. Then $B_{\Gamma'}(\alpha) = 0$.

4. If Γ has no vertex connected to three vertices, then Γ is the graph of type A_n

   Let us take a vertex $x \in \Gamma_0$. It can be connected to at most two vertices x_1 and x_{-1}. The vertices x_1 and x_{-1}. can be connected to at most two additional vertices x_2 and x_{-2} and so on. Any two x_i and x_{-i} must be pairwise different or there would be a loop and this would violate #2.

5. The graph Γ can have at most one vertex connecting to three vertices.

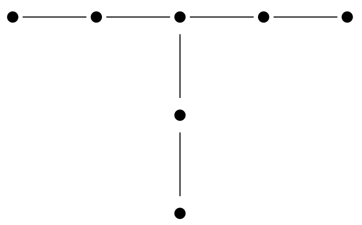
   If not, then it contains a subgraph Γ'

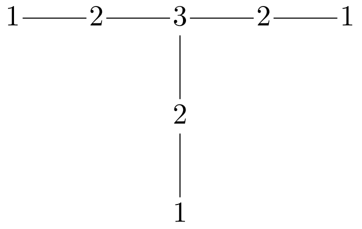
   Take the dimension vector α with values 1 on u_1, u_2, v_1, v_2 and with value 2 on the remaining vertices. Then $B_{\Gamma'}(\alpha) = 0$.

6. The graph has to be a graph of type $T_{p,q,r}$ Repeat the procedure used to prove 4. Assume $r \leq p \leq q$.

7. $r = 1$, if not the graph contains a full subgraph,

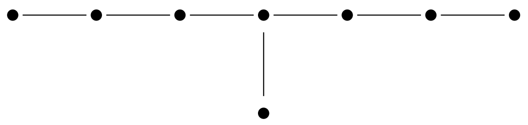


Take α to be the dimension vector

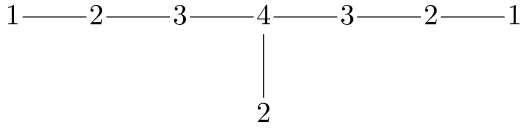


and we find $B_{\Gamma'}(\alpha) = 0$

8. $q \leq 2$, if not, Γ contains the full subgraph:



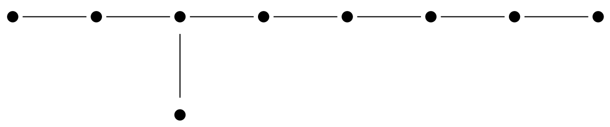
Taking the dimension vector



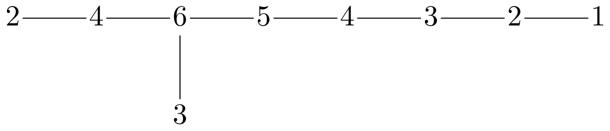
We find out that $B_{\Gamma'}(\alpha) = 0$.

9. If q = 1, then p can be arbitrary, this gives us D_n.

10. If q=2, then $p \leq 4$, if not



Taking the dimension vector



Thus this gives us only A_n, D_n, E_6, E_7 and E_8.

Proposition 2.3 *If in Q, there are only finitely many indecomposable representations, then B_Q is positive definite.*

Proof (Tits): Consider a representation V with a fixed dimension, $\underline{d}(V) = m = (m_x)$. If we fix a basis in each of the spaces $V(x)$, then V is completely determined by the set of matrices M_a, $(a \in Q_1)$, where M_a is the matrix of the mapping $V(a) : V(ta) \to V(ha)$. In each space $V(x)$

we change the basis by means of a non-singular $m_x \times m_x$ matrix g_x. Then the matrices M_a are replaced by $M'_a = g_{ha}^{-1} M_a g_{ta}$ (*)

Let M be the manifold of all sets of matrices M_a and G the group of all sets of non-singular matrices g_x. The G acts on M according to (*). Then two representations are isomorphic iff the sets of matrices $\{M_a\}$ corresponding to them lie in 1 orbit of G.

Now in Q, there are only finitely many non-isomorphic indecomposable objects, therefore the manifold M splits into a finite number of orbits of G. Now $dimG = \sum_{x \in Q_0} m_x^2$, $dimM = \sum_{a \in Q_1} m_{ta} m_{ha}$. It follows that $dimA \leq dimG - 1$, the -1 come from the fact that G has a 1-dimensional subgroup consisting of the matrices where g(ha) and g(ta) are scalar multiples of the identity.)

Therefore $-dimA + dimG - 1 \geq 0$ which impies $B_Q(m) > 0$. So by the previous proposition the graph must be one of the Dynkin quivers.

The other half of Gabriel's Theorem requires the development of the reflection functors.