SMALL SPHERICAL NILPOTENT ORBITS AND K-TYPES OF HARISH CHANDRA MODULES

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Abstract. Let G be a connected linear semisimple Lie group with Lie algebra $\mathfrak{g}$ and maximal compact subgroup K. Let $K_{\mathbb{C}} \rightarrow Aut(\mathfrak{p}_{\mathbb{C}})$ be the complexified isotropy representation at the identity coset of the corresponding symmetric space. Suppose that $\mathcal{O}$ is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$, and $\overline{\mathcal{O}}$ is its Zariski closure in $\mathfrak{p}_{\mathbb{C}}$. We study the K-type decomposition of the ring of regular functions on $\overline{\mathcal{O}}$ when $\mathcal{O}$ is "small" (e.g., $\mathcal{O}$ has height two). We also show that this decomposition gives the asymptotic directions of K-types in any irreducible $(\mathfrak{g}_{\mathbb{C}},\ K)$-module whose associated variety is $\overline{\mathcal{O}}$.

1. Introduction

Assume that G is a connected linear semisimple Lie group with maximal compact subgroup K. Let $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ denote the complexifications of G and K. If $\mathfrak{g}$ (resp., $\mathfrak{k}$) is the Lie algebra of G (resp., K), we have the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If we complexify the vector spaces in the Cartan decompostion we obtain the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$. We also obtain an action of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ by restricting the adjoint action of $G_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$.

If X is an irreducible $(\mathfrak{g}_{\mathbb{C}},\ K)$-module, then $AV(X)$, the associated variety of X, is a key invariant of X [23]. $AV(X)$ is a closed subvariety of $\mathfrak{p}_{\mathbb{C}}$ and $AV(X)$ is a finite union of nilpotent $K_{\mathbb{C}}$-orbits [23]. Many authors have studied the relationship between the K-decomposition of X and the K-decomposition of the ring of regular functions on $AV(X)$ when $AV(X)$ is the closure of a single spherical nilpotent $K_{\mathbb{C}}$-orbit. See [1] and [9]. Another example is [22] when the results are specialized to $\mathfrak{g} = so(2m, 2m)$. The special case where $AV(X)$ is the closure of a minimal non-zero nilpotent orbit has been investigated extensively. See [24].

The aim of this note is to show how the asymptotic behavior of the K-types of X can be described precisely when $AV(X)$ is the closure of a nilpotent $K_{\mathbb{C}}$-orbit which is small (Definition 3.2) and spherical (Definition 3.1). The main result is Theorem 4.2. This result hinges on two pleasant properties of the closure of a small spherical nilpotent $K_{\mathbb{C}}$-orbit. The first is that the ring of regular functions on the Zariski closure of the orbit is a normal ring. The second is that the highest weight vectors for the irreducible representations of K in this ring form a polynomial algebra. This is the main content of Proposition 3.1.

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2. A DESINGULARISATION OF THE CLOSURE OF A NILPOTENT $K_{\mathbb{C}}$-ORBIT IN $\mathfrak{p}_{\mathbb{C}}$

Suppose that $\mathcal{O}$ is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$, and $\overline{\mathcal{O}}$ is its Zariski closure in $\mathfrak{p}_{\mathbb{C}}$. $\overline{\mathcal{O}}^n$ denotes the normalization of $\overline{\mathcal{O}}$. If W is an algebraic variety, $R[W]$ denotes the ring of regular functions on W.

We first recall some facts related to vector bundles over a homogeneous space $M \ / \ P$ for a complex reductive group M and parabolic subgroup P. Let V be a finite dimensional P-module. Then, $M \times_P V = M \times V / \sim$, where the equivalence relation "$\sim$" is defined by: $(gy,\ v) \sim (g,\ y \cdot v)$, for $g \in M$, $y \in P$ and $v \in V$. The equivalence relation "$\sim$" can be seen as arising from a right action of P on $M \times V$ defined as follows: $(g,\ v) \cdot y = (gy,\ y^{-1} \cdot v)$. Then it is easy to check that $(g,\ v) \cdot (y_1 y_2) = ((g,\ v) \cdot y_1) \cdot y_2$. Also, note that $(g,\ v) \sim (g,\ v) \cdot y$, for all $g \in M,\ v \in V,\ y \in P$. The equivalence class of $(g,\ v)$ under $\sim$ is denoted $[g,\ v]$.

A key fact is that:

$$R[M \times_P V] = R[M \times V]^P,$$

where the P action on $R[M \times V]$ is defined from the (right) P action on $M \times V$: if $k \in R[M \times V]$, and $y \in P$, then $(k \cdot y)(g,\ v) = k((g,\ v) \cdot y^{-1}) = k(gy^{-1},\ y \cdot v)$. The equality of the rings above is implemented as follows. If $f \in R[M \times_P V]$, then define $\widetilde{f}$ on $M \times V$, by $\widetilde{f}(g,\ v) = f([g,\ v])$. Since $f([g,\ v]) = f([gyy^{-1},\ v]) = f([gy,\ y^{-1} \cdot v])$, $\widetilde{f} \in R[M \times V]^P$. On the other hand if $h \in R[M \times V]^P$, then $\hat{h}([g,\ v]) := h(g,\ v)$ is well defined and $\hat{h} \in R[M \times_P V]$.

Furthermore,

$$\begin{aligned} R[M \times V]^P &= (R[M] \otimes_{\mathbb{C}} R[V])^P = (R[M] \otimes_{\mathbb{C}} S[V^*])^P \\ &= \sum_{n \geq 0} (R[M] \otimes_{\mathbb{C}} S^n[V^*])^P = \sum_{n \geq 0} H^0(M/P,\ S^n(V^*)), \end{aligned}$$

where $S(V^*)$ is the symmetric algebra on V^*, the dual of V. Here, the action of P on $R[M] \otimes_{\mathbb{C}} R[V]$ is: $[(f_1(\cdot) \otimes f_2(\cdot)) \cdot y](g,\ v) = f_1(gy^{-1}) f_2(y \cdot v)$

Let $\{x,\ e,\ f\}$ be a normal triple, in the sense of Kostant and Rallis [5], corresponding to the nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{p}_{\mathbb{C}}$.

Definition 2.1. *(The eigenspaces of $ad(x)$ on $\mathfrak{g}_{\mathbb{C}}$) Let* $\mathfrak{g}_{\mathbb{C}}(x;j)$ = *the j-eigenspace of x. Likewise define $\mathfrak{p}_{\mathbb{C}}(x;j)$ and $\mathfrak{k}_{\mathbb{C}}(x;j)$. $\mathcal{O}$ will be said to have height j, if $\mathfrak{g}_{\mathbb{C}}(x;j) \neq 0$ but $\mathfrak{g}_{\mathbb{C}}(x;j') = 0$ for all $j' > j$.*

Definition 2.2. *(Construction of a desingularization of $\overline{\mathcal{O}}$.) Set $V = V(e) = \sum_{j \geq 2} \mathfrak{g}_{\mathbb{C}}(x;j)$. $\mathfrak{q}_{\mathbb{C}} = \sum_{j \geq 0} \mathfrak{g}_{\mathbb{C}}(x;j)$, $\mathfrak{u}_{\mathbb{C}} = \sum_{j > 0} \mathfrak{g}_{\mathbb{C}}(x;j)$, and $\mathfrak{l}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}(x;0)$. Let $\mathbf{Q}_{\mathbb{C}}$, $L_{\mathbb{C}}$, and $U_{\mathbb{C}}$ be the connected subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{q}_{\mathbb{C}}$, $\mathfrak{l}_{\mathbb{C}}$, and $\mathfrak{u}_{\mathbb{C}}$ respectively. It is well known that the morphism $\pi : G_{\mathbb{C}} \times_{\mathbf{Q}_{\mathbb{C}}} V \to \overline{G_{\mathbb{C}} \cdot e}$, defined by $\pi([g,v)] = g \cdot v$ is a desingularization of $\overline{G_{\mathbb{C}} \cdot e}$. By similar arguments, if $\widetilde{V} = \sum_{j \geq 2} \mathfrak{p}_{\mathbb{C}}(x;j) = V \cap \mathfrak{p}_{\mathbb{C}}$, then the (restriction) mapping*

$$\pi : K_{\mathbb{C}} \times_{\mathbf{Q}_{\mathbb{C}} \cap K_{\mathbb{C}}} \widetilde{V} \to \overline{K_{\mathbb{C}} \cdot e} \tag{1}$$

is a desingularization (resolution of singularities) of $\overline{\mathcal{O}}$.

Remark 2.1. *Since* $ad(e) : \mathfrak{q}_{\mathbb{C}} \to V$ *is surjective,* $ad(e) : \mathfrak{q}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}} \to V \cap \mathfrak{p}_{\mathbb{C}}$ *is surjective. That is,* $[\mathfrak{q}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}},\ e] = V \cap \mathfrak{p}_{\mathbb{C}}$. *This enables us to conclude that the orbit* $Q \cap K_{\mathbb{C}} \cdot e$ *is dense in* $\widetilde{V}$. *The remainder of the argument is essentially the same as the one establishing that* $\pi : G_{\mathbb{C}} \times_{\mathbf{Q}_{\mathbb{C}}} V \to \overline{G_{\mathbb{C}} \cdot e}$ *is a desingularization of* $\overline{G_{\mathbb{C}} \cdot e}$.*)*

Proposition 2.1. *With notation as above,* $R[K_{\mathbb{C}} \times_{\mathbf{Q}_{\mathbb{C}} \cap K_{\mathbb{C}}} \widetilde{V}] = R[\overline{\mathcal{O}}^n]$

Proof. Set $X = K_{\mathbb{C}} \times_{\mathbf{Q}_{\mathbb{C}} \cap K_{\mathbb{C}}} \widetilde{V}$. We have that

$$R[\overline{\mathcal{O}}] \subset R[\overline{\mathcal{O}}^n] \subset R[X] \subset R[\mathcal{O}].$$

(The last containment follows because $\pi^{-1}(\mathcal{O})$ can be identified with $\mathcal{O}$.) Moreover, if A is an integral domain let $\mathcal{Q}(A)$ denote the field of fractions of A, and if W is an algebraic variety, let $\mathcal{F}(W)$ denote the function field of W. Then,

$$R[X] \subset \mathcal{Q}(R[X]) \subset \mathcal{F}(X).$$

Since π is birational, $\mathcal{F}(X) = \mathcal{F}(\overline{\mathcal{O}})$. But, since $\mathcal{O}$ is affine, $\mathcal{F}(\overline{\mathcal{O}}) = \mathcal{Q}\left(R[\overline{\mathcal{O}}]\right)$. We also know that $R[X]$ is finitely generated over $R[\overline{\mathcal{O}}]$. Since $R[\overline{\mathcal{O}}]$ is a Noetherian ring, $R[X]$ is a noetherian $R[\overline{\mathcal{O}}]$-module. Now suppose $f \in R[X]$, then $R[\overline{\mathcal{O}}][f]$ the polynomial ring in the variable f with coefficients in $R[\overline{\mathcal{O}}]$, is an $R[\overline{\mathcal{O}}]$ submodule of $R[X]$. Hence $R[\overline{\mathcal{O}}][f]$ is finitely generated over $R[\overline{\mathcal{O}}]$. But then f is integral over $R[\overline{\mathcal{O}}]$ which implies that $f \in R[\overline{\mathcal{O}}^n]$, since $R[\overline{\mathcal{O}}^n]$ is the integral closure of $R[\overline{\mathcal{O}}]$ in $\mathcal{Q}\left(R[\overline{\mathcal{O}}]\right)$. This implies the $R[X] \subset R[\overline{\mathcal{O}}^n]$ and so the rings are equal. ☐

Corollary 2.1. *If* $\mathcal{O}$ *is a nilpotent* $K_{\mathbb{C}}$ *orbit in* $\mathfrak{p}_{\mathbb{C}}$ *then*

$$R[\overline{\mathcal{O}}] \subset R[\overline{\mathcal{O}}^n] \subset R[\mathcal{O}].$$

Corollary 2.2. *With notation as in Proposition 2.1, if* $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ *then* $\overline{\mathcal{O}}$ *is normal.*

Even if $\overline{\mathcal{O}}$ is normal, we don't necessarily have $R[\overline{\mathcal{O}}] = R[\mathcal{O}]$. In case the codimension of the boundary of $\overline{\mathcal{O}}$ in $\overline{\mathcal{O}}$ is at least 2, we have the following observation of Kostant.

Proposition 2.2. *Let* $S(\mathcal{O})$ *denote the set of restrictions of polynomials on* $\mathfrak{p}_{\mathbb{C}}$ *to* $\mathcal{O}$. *If* $\overline{\mathcal{O}}$ *is normal, and the codimension of the complement of* $\mathcal{O}$ *in* $\overline{\mathcal{O}}$ *is at least two, then* $S(\mathcal{O}) = R[\mathcal{O}]$*, and hence* $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$.

Proof. See Proposition 9 in [15]. ☐

3. NORMALITY OF CLOSURES OF SMALL SPHERICAL ORBITS

Definition 3.1. *The nilpotent orbit* $\mathcal{O} = K_{\mathbb{C}} \cdot e$ *in* $\mathfrak{p}_{\mathbb{C}}$ *is said to be spherical if some Borel subgroup of* $K_{\mathbb{C}}$ *has a dense orbit in* $\mathcal{O}$.

A classification of spherical nilpotent orbits for $\mathfrak{g}$ real and simple may be found in [12].

Definition 3.2. *A nilpotent* $K_{\mathbb{C}}$*-orbit* $\mathcal{O}$ *in* $\mathfrak{p}_{\mathbb{C}}$ *is said to be small if* $\mathfrak{p}_{\mathbb{C}}(x; i) = 0$ *for all* $i > 2$.

Remark 3.1. *If an orbit $\mathcal{O}$ satisfies $\mathfrak{p}_{\mathbb{C}}(x;i)=0$ for $i>2$, we have the exact sequence:*

$$0 \to {\mathfrak{k}_{\mathbb{C}}}^{\{x,\ e,\ f\}} \to \mathfrak{k}_{\mathbb{C}}^x \to \mathfrak{p}_{\mathbb{C}}(x;2) \to 0,$$

where $ad(e)$ is the map $\mathfrak{k}_{\mathbb{C}}^x \to \mathfrak{p}_{\mathbb{C}}(x;2)$.

Remark 3.2. *$\mathcal{O}$ may be small but not spherical. For example, for $\mathfrak{g}=su(6,\ 3)$ there is a unique orbit $\mathcal{O}$ corresponding to the partition $3+3+3$ of 9. This orbit is small. However, the dimension of a Borel subgroup of $K_{\mathbb{C}}$ is 26, while the dimension of $\mathcal{O}$ is 27, so that $\mathcal{O}$ is not spherical. (See* [5] *for an explanation of the parametrization of the nilpotent orbits of $su(p,\ q)$.)*

Remark 3.3. *If $\mathcal{O}$ is spherical and small, we may have $\mathfrak{k}_{\mathbb{C}}(x;4)\neq 0$. The principal nilpotent orbit of $su(2,\ 1)$ satisfies these three conditions.*

Remark 3.4. *Suppose that $\mathcal{O}$ has height 2. Then $\mathcal{O}$ is spherical and small.*

Remark 3.5. *Suppose $\mathcal{O}$ is spherical and small, then $\widetilde{V}$ (see Definition 2.2) is a commutative subspace of $\mathfrak{p}_{\mathbb{C}}$ if and only if $\mathcal{O}$ has height* 2.

Let T be a maximal torus in K whose Lie algebra contains ix. Let us assume that we have chosen a positive system Δ_k^+ for $(\mathfrak{k}_{\mathbb{C}},\ \mathfrak{t}_{\mathbb{C}})$ so that x is dominant with respect to Δ_k^+. $\mathfrak{n}_k$ is the nilradical of $\mathfrak{k}_{\mathbb{C}}$ that corresponds to Δ_k^+ and ${\mathfrak{n}_k}^-$ is the opposite nilradical. Set $u(\mathfrak{l}_k)=\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{k}_{\mathbb{C}}\cap\mathfrak{n}_k$. Then, we may assume that $u(\mathfrak{l}_k)$ is the nilradical of $\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{k}_{\mathbb{C}}$, and $u(\mathfrak{l}_k)^-$ denotes the opposite nilradical of $\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{k}_{\mathbb{C}}$.

Definition 3.3. *If $\mathcal{O}$ is a $K_{\mathbb{C}}$-nilpotent orbit in $\mathfrak{p}_{\mathbb{C}}$, and $\lambda\in\widehat{K}$, the set of equivalence classes of irreducible representations of K, let $m_\lambda(\mathcal{O})$ (resp., $m_\lambda(\overline{\mathcal{O}})$) denote the multiplicity of λ in $R[\mathcal{O}]$ (resp., $R[\overline{\mathcal{O}}]$).*

Definition 3.4. *We say that $R[\overline{\mathcal{O}}]$ is self dual as a K-module if for every K-type $\lambda\in\widehat{K}$, $m_\lambda(\overline{\mathcal{O}})=m_{\lambda^*}(\overline{\mathcal{O}})$ where $\lambda^*\in\widehat{K}$ is the dual of λ.*

Definition 3.5. *(Weyl involution of $\mathfrak{g}_{\mathbb{C}}$) Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing $\mathfrak{t}_{\mathbb{C}}$. Let $\Delta=\Delta(\mathfrak{g}_{\mathbb{C}},\ \mathfrak{h})$, be the corresponding set of roots. Δ^+ is a positive system for Δ. Assume that we have a Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$: $\{H_\alpha,\ X_\alpha,\ X_{-\alpha}|\ \alpha\in\Delta^+\}$. Then, ν, the Weyl involution of $\mathfrak{g}_{\mathbb{C}}$ is defined by the requirement that $\nu(H_\alpha)=-H_\alpha$, and $\nu(X_\alpha)=-X_{-\alpha}$ for all $\alpha\in\Delta^+$.*

Let w_0 be an element in the normalizer of T in K that represents the longest element of the Weyl group of $(K,\ T)$.

Proposition 3.1. *(Hesselink, Panyushev) As above let $\mathcal{O}=K_{\mathbb{C}}\cdot e$ be spherical and let the corresponding triple be $\{x,\ e,\ f\}$. Assume that $\mathcal{O}$ is small. Then (1) $\overline{\mathcal{O}}$ is normal; (2) $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ is a polynomial algebra; (3) If $R[\overline{\mathcal{O}}]$ is self dual then there is a T-equivariant ring isomorphism between $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ and $S[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)}$. (4) If $R[\overline{\mathcal{O}}]$ is not self dual, a set of generators of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ can be identified with elements in $S(w_0\cdot(\nu(\mathfrak{p}_{\mathbb{C}}(x;2))))$.*

Proof. This depends on ideas of Hesselink (based on work of Kempf) applied to the desingularisation of $\overline{\mathcal{O}}$ in Definition 2.2 above. Since $\mathfrak{p}_{\mathbb{C}}(x;i)=0$ if $i>2$, $\widetilde{V}=\mathfrak{p}_{\mathbb{C}}(x;2)$, and the map in (1) can be rewritten:

$$\pi : K_{\mathbb{C}}\times_{\mathbf{Q}_{\mathbb{C}}\cap K_{\mathbb{C}}}\mathfrak{p}_{\mathbb{C}}(x;2)\to\overline{\mathcal{O}}=K_{\mathbb{C}}\cdot\mathfrak{p}_{\mathbb{C}}(x;2). \tag{2}$$

Since $\mathfrak{p}_{\mathbb{C}}(x;i)=0$ if $i>2$, the nilradical of $\mathbf{Q}_{\mathbb{C}}\cap K_{\mathbb{C}}$ acts trivially on $\mathfrak{p}_{\mathbb{C}}(x;2)$. If we decompose $\mathfrak{p}_{\mathbb{C}}(x:2)$ into irreducible $L\cap K_{\mathbb{C}}$ modules then each of these modules will be irreducible under $\mathbf{Q}_{\mathbb{C}}\cap K_{\mathbb{C}}$. Thus $\mathfrak{p}_{\mathbb{C}}(x;2)$ is completely reducible as a $\mathbf{Q}_{\mathbb{C}}\cap K_{\mathbb{C}}$ module. From Proposition 2.1, we know that $R[K_{\mathbb{C}}\times_{\mathbf{Q}_{\mathbb{C}}\cap K_{\mathbb{C}}}\mathfrak{p}_{\mathbb{C}}(x;2)]$ is the ring of functions of the normalization and that $R[\overline{\mathcal{O}}]\subset R[K_{\mathbb{C}}\times_{\mathbf{Q}_{\mathbb{C}}\cap K_{\mathbb{C}}}\mathfrak{p}_{\mathbb{C}}(x;2)]$. Lemma 5.5 of [18] (based on ideas of Kempf) implies that:

$$R[K_{\mathbb{C}}\cdot\mathfrak{p}_{\mathbb{C}}(x;2)]^{{\mathfrak{n}_k}^-}=R[K_{\mathbb{C}}\times_{\mathbf{Q}_{\mathbb{C}}\cap K_{\mathbb{C}}}\mathfrak{p}_{\mathbb{C}}(x;2)]^{{\mathfrak{n}_k}^-} \tag{3}$$

and there is a T equivariant ring homomorphism

$$R[\overline{\mathcal{O}}]^{(\mathfrak{n}_k)^-}\to R[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)^-}. \tag{4}$$

Equation (3) implies that $R[\overline{\mathcal{O}}]=R[K_{\mathbb{C}}\times_{\mathbf{Q}_{\mathbb{C}}\cap K_{\mathbb{C}}}\mathfrak{p}_{\mathbb{C}}(x;2)]$ and hence $\overline{\mathcal{O}}$ is normal.

$R[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)^-}$ is a polynomial algebra because $\mathfrak{p}_{\mathbb{C}}(x;2)$ is a spherical $\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{k}_{\mathbb{C}}$-module. (See Corollary 12.2.5 in [8].)

$R[\mathfrak{p}_{\mathbb{C}}(x;2)]$ can be identified with the symmetric algebra $S[\mathfrak{p}_{\mathbb{C}}(x;2)^*]$, where $\mathfrak{p}_{\mathbb{C}}(x;2)^*$ is the dual of $\mathfrak{p}_{\mathbb{C}}(x;2)$. But $\mathfrak{p}_{\mathbb{C}}(x;2)^*$ can be identified with the space $\mathfrak{p}_{\mathbb{C}}(x;-2)$. The Weyl involution ν maps $\mathfrak{p}_{\mathbb{C}}(x;-2)$ to $\mathfrak{p}_{\mathbb{C}}(x;2)$. Moreover, this linear isomorphism extends to a ring isomorphism of $S(\mathfrak{p}_{\mathbb{C}}(x;-2)$ with $S(\mathfrak{p}_{\mathbb{C}}(x;2)$. This ring isomorphism restricts to a ring isomorphism:

$$S(\mathfrak{p}_{\mathbb{C}}(x;-2)^{u(\mathfrak{l}_k)^-}\to S(\mathfrak{p}_{\mathbb{C}}(x;2)^{u(\mathfrak{l}_k)}. \tag{5}$$

The preceding isomorphism carries a lowest weight vector of weight λ to a highest weight vector of weight $-\lambda$.

The element w_0 defines a ring isomorphism $R[\overline{\mathcal{O}}]^{(\mathfrak{n}_k)^-}\to R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$. This fact together with the isomorphisms (4) and (5) imply that $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ is a polynomial ring.

Now assume $R[\overline{\mathcal{O}}]$ is self dual, then by normalizing the highest weight and lowest weight vectors in each irreducible K submodule, we can define a ring isomorphism:

$$R[\overline{\mathcal{O}}]^{{\mathfrak{n}_k}^-}\to R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}. \tag{6}$$

The isomorphism takes a lowest weight vector of weight $w_0\lambda$ to a highest weight vector of weight $-w_0\lambda$ in the dual representation. Combining isomorphisms (4), (5) and (6), we obtain a T-equivariant ring isomorphism

$$R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}\to S[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)}.$$

and thus generators for $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ lie in $S[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)}$.

When $R[\overline{\mathcal{O}}]$ is not self dual, it is clear that we can find generators of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ in $S(w_0\cdot(\mathfrak{p}_{\mathbb{C}}(x;-2)))$. □

Remark 3.6. *$\overline{\mathcal{O}}$ may fail to be normal when $\mathcal{O}$ is spherical but not small. J. Weyman (private communication) has shown that the closure of either spherical nilpotent orbit associated to the Lie algebra $so(4,\ 3)$ and the partition $3+2+2$ is not normal.*

Corollary 3.1. *If $\mathcal{O}$ is a $K_{\mathbb{C}}$-nilpotent orbit in $\mathfrak{p}_{\mathbb{C}}$, and $\mathcal{O}$ is small and spherical, then $R[\overline{\mathcal{O}}]$ is self dual if and only if $-w_0$ preserves the set of highest weights of the generators of $S[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)}$.*

Proof. This follows from the proof of Proposition 3.1. □

Remark 3.7. *Many authors have studied the K-decomposition of $R[\overline{\mathcal{O}}]$ when $\mathcal{O}$ is spherical. For example, in* [3] *Binegar has determined the K-decomposition of many height two spherical nilpotent orbits when $\mathfrak{g}$ is simple and classical. His results are consistent with Proposition 3.1, and allow him to give a formula for the degree of $\overline{\mathcal{O}}$ for many height two orbits. Nishiyama, Ochiai and Zhu have also studied the K-decomposition of $R[\overline{\mathcal{O}}]$ for small spherical orbits using the method of "theta-lifting." See* [17].

In order to use Proposition 3.1 to find the highest weights of the generators for $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$, we first find the set of highest weights for the generators of $S[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)}$ and then apply $-w_0$ to this set of highest weights.

Definition 3.6. *If $\mathcal{O}$ is a $K_{\mathbb{C}}$-nilpotent orbit in $\mathfrak{p}_{\mathbb{C}}$, then*

$$\Gamma(\overline{\mathcal{O}}) := \{\mu \in \widehat{K} | m_{\overline{\mathcal{O}}}(\mu) \neq 0\}$$

Similarly define $\Gamma(\mathcal{O}) =: \{\mu \in \widehat{K} | m_{\mathcal{O}}(\mu) \neq 0\}$.

$\Gamma(\overline{\mathcal{O}})$ and $\Gamma(\mathcal{O})$ are finitely generated semigroups inside the semigroup of dominant weights for $(\mathfrak{k},\ \mathfrak{t})$ in $\mathfrak{t}_{\mathbb{C}}^* = Hom(\mathfrak{t}_{\mathbb{C}},\ \mathbb{C})$.

Definition 3.7. *(See* [19]*) Let B_k be a Borel subgroup of $K_{\mathbb{C}}$ and let N_k denote its nilradical. ($\mathfrak{n}_k$ is the Lie algebra of N_k.) If $\mathcal{O}$ is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$, the $K_{\mathbb{C}}$-rank of $\mathcal{O}$ (resp., $\overline{\mathcal{O}}$), denoted $r_{\mathcal{O}}$ (resp., $r_{\overline{\mathcal{O}}}$), is equal to the transcendence degree of the field of N_k-invariant rational functions on $\mathcal{O}$ (resp., $\overline{\mathcal{O}}$), which is the same as the codimension of a generic N_k orbit in $\mathcal{O}$ (resp., $\overline{\mathcal{O}}$). Thus, it is clear that $r_{\mathcal{O}}$ equals $r_{\overline{\mathcal{O}}}$.*

Example 3.1. *If $\mathfrak{g}$ is simple, and $\mathcal{O}$ is a minimal non zero nilpotent orbit, then $r_{\mathcal{O}} = 1$.*

Remark 3.8. *Suppose that $\mathcal{O}$ is spherical. Since $\mathcal{O}$ and $\overline{\mathcal{O}}$ are quasiaffine, it follows that $r_{\mathcal{O}}$ is equal to the Krull dimension of $R[\mathcal{O}]^{\mathfrak{n}_k}$ and $r_{\overline{\mathcal{O}}}$ is equal to the Krull dimension of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$. See the proof of Theorem 1 in* [21]. *Moreover, the Krull dimension of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ is equal to the rank of $\Gamma(\overline{\mathcal{O}})$, and the Krull dimension of $R[\mathcal{O}]^{\mathfrak{n}_k}$ is equal to the rank of $\Gamma(\mathcal{O})$.*

Corollary 3.2. *Let $\mathcal{O}$ be a $K_{\mathbb{C}}$-nilpotent orbit in $\mathfrak{p}_{\mathbb{C}}$ which is spherical and small. Then, the number of algebraically independent generators of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ is equal to $r_{\mathcal{O}}$.*

Proof. Combine Definition 3.7 and Remark 3.8.

□

Definition 3.8. *Let $\mathcal{O}$ be a $K_{\mathbb{C}}$-nilpotent orbit in $\mathfrak{p}_{\mathbb{C}}$ which is spherical and small. Set $r = r_{\mathcal{O}}$. Since $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ is a polynomial ring, by Proposition 3.1, we can find homogeneous generators $f_1, \ldots,\ f_r$ of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$. Furthermore, if $R[\overline{\mathcal{O}}]$ is self dual, the generators can be assumed to be irreducible elements of $S[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)}$. Let the degree of f_i be denoted by d_i and its $\mathfrak{t}$-weight be denoted by γ_i. Thus, for each $\mathcal{O}$, the set of pairs $\{(d_i,\ \gamma_i)\}$ is unique.*

Example 3.2. $\mathfrak{g} = sp(n,\ \mathbf{R})$. *In (8) and (9) below, we give the set $\{(d_i,\ \gamma_i)\}$ for each spherical orbit. For $\mathfrak{g} = sp(n,\ \mathbf{R})$, all spherical orbits are small. In fact, each non-zero spherical orbit has height two. They are parametrized by signed Young tableaux with $2n$ boxes in which plus signs and minus signs alternate in each* *row. See* [12]. *For example for $n = 2$, the non trivial spherical nilpotent orbits are represented by:*

$$\begin{array}{ll}+&-\\+&-\end{array},\quad \begin{array}{ll}+&-\\-&+\end{array},\quad \begin{array}{ll}-&+\\-&+\end{array},\quad \begin{array}{ll}+&-\\+&\\+&\end{array},\quad \begin{array}{ll}-&+\\+&\\+&\end{array}.$$

We abbreviate the previous diagrams as $(+2)^2$, $(+2)(-2)$, $(-2)^2$, $(+2)(+1)^2$ and $(-2)(+1)^2)$. A general spherical nilpotent orbit $\mathcal{O}$ of $sp(n, \mathbf{R})$ is represented by the expression: $(+2)^{k_1}(-2)^{k_2}(+1)^{2n-2k_1-2k_2}$. If the zero orbit is included, there are $(1/2)(n+2)(n+1)$ such orbits. Adopt the notation of appendix C of [13] *regarding $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}}$ and the roots of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{t}_{\mathbb{C}}$, the complexification of a fundamental Cartan subalgebra $\mathfrak{t}$. Choose the positive system given by the Vogan diagram. Thus all simple roots are compact except for $2e_n$ which labels the right most node in the Vogan diagram. We choose the positive system for $(\mathfrak{k}_{\mathbb{C}},\ \mathfrak{t}_{\mathbb{C}})$ whose simple roots are $\{e_1 - e_2,\ \ldots,\ e_{n-1} - e_n\}$. Set $\mathcal{O} = (+2)^{k_1}(-2)^{k_2}(+1)^{2n-2k_1-2k_2}$, with $\{x,\ e,\ f\}$ a corresponding normal triple. Let $\mathfrak{s}$ be the subalgebra spanned by $\{x,\ e,\ f\}$. By applying Proposition 5.8 in* [12], *one can determine the neutral element x, and hence $\mathfrak{k}^x$. We have*

$$\mathfrak{k}^x = u(k_1) \oplus u(n - k_1 - k_2) \oplus u(k_2). \tag{7}$$

The simple roots for the summands of $\mathfrak{k}^x$ are as follows:

$$\begin{aligned} &u(k_1):\ \{e_1 - e_2,\ \ldots,\ e_{k_1-1} - e_{k_1}\}\\ &u(n - k_1 - k_2):\ \{e_{k_1+1} - e_{k_1+2},\ \ldots,\ e_{n-k_2-1} - e_{n-k_2}\}\\ &u(k_2):\ \{e_{n-k_2+1} - e_{n-k_2+2},\ \ldots,\ e_{n-1} - e_n\}\end{aligned}$$

Let W_1 be the span of the root vectors corresponding to the noncompact roots

$$\{2e_1,\ \ldots, 2e_{k_1}\} \cup \{e_1 + e_2,\ \ldots,\ e_{k_1-1} + e_{k_1}\},$$

and W_2 be the span of the root vectors corresponding to the noncompact roots

$$\{-2e_{n-k_2+1},\ \ldots,\ -2e_n\} \cup \{-e_{n-k_2+1} - e_{n-k_2+2},\ \ldots,\ -e_{n-1} - e_n\}.$$

Then, again by applying Proposition 5.8 of [12], *we have $\mathfrak{p}_{\mathbb{C}}(x; 2) = W_1 \oplus W_2$. Since $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$ is the complexification of $\mathfrak{k}^x$, the $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$-module structure of $\mathfrak{p}_{\mathbb{C}}(x : 2)$ is the same as its $\mathfrak{k}^x$-module structure. The action of $u(k_1)$ summand of $\mathfrak{k}^x$ on W_1 is the representation $u(k_1)$ on $S^2(\mathbb{C}^{k_1})$. (The highest weight is $2e_1$.) This summand acts trivially on W_2. The action of the $u(k_2)$ summand of $\mathfrak{k}^x$ on W_2 is the representation of $u(k_2)$ on $S^2((\mathbb{C}^{k_2})^*)$. (The highest weight is $-2e_n$.) This summand acts trivially on W_1. The $u(n - k_1 - k_2)$ summand of $\mathfrak{k}^x$ acts trivially on W_1 and W_2.*

The decomposition of the symmetric algebra of $S^2(\mathbb{C}^{k_1})$ (resp., $S^2((\mathbb{C}^{k_2})^*)$) under $u(k_1)$ (resp., $u(k_2)$) is well known. Thus, if Λ_1 denotes the set of highest weights of the generators of the highest weights of the symmetric algebra of W_1 under the action of $u(k_1)$, then

$$\Lambda_1 = \{2e_1,\ 2e_1 + 2e_2,\ \ldots,\ 2e_1 + 2e_2 + \ldots + 2e_{k_1}\}.$$

The degrees of the corresponding generators are $1, \ldots, k_1$. If Λ_2 denotes the set of highest weights of the generators of the highest weights of the symmetric algebra of W_2 under the action of $u(k_2)$, then

$$\Lambda_2 = \{-2e_n,\ -2e_n - 2e_{n-1},\ \ldots,\ -2e_n - 2e_{n-1} - \ldots - 2e_{n-k_2+1}\}.$$

The degrees of the corresponding generators are $1, \ldots, k_2$. Since $S(\mathfrak{p}_{\mathbb{C}}(x; 2)) = S(W_1) \otimes S(W_2)$, if Λ is the set of highest weights of the generators of $S[\mathfrak{p}_{\mathbb{C}}(x; 2)]^{u(\mathfrak{l}_k)}$, then $\Lambda = \Lambda_1 \cup \Lambda_2$.

The set $-w_0\Lambda$ *is the set of highest weights of the generators of* $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ *and is the union of:*

$$\{2e_1,\ 2e_1+2e_2,\ \ldots,\ 2e_1+2e_2+\ldots+2e_{k_2}\} \tag{8}$$

and

$$\{-2e_n,\ -2e_n-2e_{n-1},\ \ldots,\ -2e_n-2e_{n-1}-\ldots-2e_{n-k_1+1}\}. \tag{9}$$

The degrees of the generators of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ *with weights in (8) are* $1,\ldots,\ k_1$ *and the degrees of the generators with weights in (9) are* $1,\ldots,\ k_2$*.*

If $k_1=0$ *or* $k_2=0$*, our results agree with those of Binegar in* [3]*.*

By corollary 3.3 in [20]*,* $r_{\mathcal{O}}$ *is equal to the rank of* $K^x/K^{\mathfrak{s}}$*. Since* $\mathfrak{k}^x$ *has the form given above in (7) and* $\mathfrak{k}^{\mathfrak{s}}=so(k_1)\oplus u(n-k_1-k_2)\oplus so(k_2)$*, we obtain*

$$r_{\mathcal{O}}=k_1+k_2, \tag{10}$$

which agrees with the cardinality of the set of weights of the generators in (8) and (9).

The author is currently computing the set $\{d_i,\ \gamma_i)$ for each small spherical orbit when $\mathfrak{g}$ is real, classical and simple. In section 5, we tabulate this information for the real exceptional simple algebras.

The author is not aware of any published work which lists, for each simple $\mathfrak{g}$, all the nilpotent $K_{\mathbb{C}}$-orbits $\mathcal{O}$ in $\mathfrak{p}_{\mathbb{C}}$ such that $R[\overline{\mathcal{O}}]$ is self dual. Nor have all the spherical nilpotent $K_{\mathbb{C}}$-orbits been determined for which $R[\overline{\mathcal{O}}]$ is self dual. If $\mathfrak{k}$ is semisimple, and all irreducible representations of $\mathfrak{k}$ are self-dual, then clearly each $R[\overline{\mathcal{O}}]$ is self dual. Thus, $R[\overline{\mathcal{O}}]$ is self dual for each nilpotent $K_{\mathbb{C}}$-orbit of each of the following simple Lie algebras: $su^*(2n)$, $sp(p,\ q)$, EI, EIV, EVI, $EVIII$, EIX, FI, FII, and the split real form of G_2.

Remark 3.9. *For* $\mathfrak{g}=sl(n,\ \mathbf{R})$*,* $R[\overline{\mathcal{O}}]$ *is self dual for all spherical orbits* $\mathcal{O}$ *except when* $n=2m$*, and* m *is odd. In that case,* $R[\overline{\mathcal{O}}]$ *is not self-dual if* $\mathcal{O}$ *is either of the orbits corresponding to the partition* 2^m*. (*2^m *is the partition of* n *in which each part size equals* 2*.) But* $R[\overline{\mathcal{O}}]$ *is self dual for all other spherical orbits.*

4. Applications to Harish Chandra modules

Throughout this section assume that $\mathcal{O}$ is a small, spherical nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$. $\{x,\ e,\ f\}$ is a normal triple corresponding to $\mathcal{O}$.

Definition 4.1. $\mathbf{R}^+\left(\Gamma(\overline{\mathcal{O}})\right)$ *denotes the cone over* $\mathbf{R}^+$ *generated by* $\Gamma(\overline{\mathcal{O}})$ *(See Definition 3.6.)*

As in Definition 3.8, let $f_i=f_{i,\ e}$, $i=1,\ldots,\ r=r(\mathcal{O})$, be the generators of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ and let the $\mathfrak{t}$ weight of f_i be γ_i. We assume that either for each i, $f_i\in S(\mathfrak{p}_{\mathbb{C}}(x;2))$ or, for each i, $f_i\in S(w_0\cdot(\mathfrak{p}_{\mathbb{C}}(x;-2)))$. (Recall that w_0 is an element of K corresponding to the longest element of the Weyl group of K.) If $\sigma: S(\mathfrak{g}_{\mathbb{C}})\to U(\mathfrak{g}_{\mathbb{C}})$ is the symmetrization map, set

$$u_i:=\sigma(f_i). \tag{11}$$

Definition 4.2. *(See* [23]*.) Let* X *be an irreducible* $(\mathfrak{g}_{\mathbb{C}},\ K)$*-module. Suppose that* M_0 *is a* K*-stable finite dimensional subspace of* X*.*

Set $M_i = U_i(\mathfrak{g}_{\mathbb{C}})M_0$*, where* $U_i(\mathfrak{g}_{\mathbb{C}})$ *is the standard increasing filtration of* $U(\mathfrak{g}_{\mathbb{C}})$*. Define the associated graded module* $gr(X)$ *as follows:*

$$gr(X) := gr(X; M_0) = M_0/\{0\} \oplus M_1/M_0 \oplus \ldots \oplus M_i/M_{i-1} \oplus \ldots . \tag{12}$$

$gr(X)$ *is a module for* $S(\mathfrak{g}_{\mathbb{C}})$*.* $AV(X)$*, the associated variety of* X *is defined to be the zero set in* $\mathfrak{g}^*_{\mathbb{C}}$ *of the annihilator of* $gr(X)$ *in* $S(\mathfrak{g}_{\mathbb{C}})$*.* $AV(X)$ *is independent of the choice of* M_0*. By identifying* $\mathfrak{g}_{\mathbb{C}}$ *with* $\mathfrak{g}^*_{\mathbb{C}}$ *using the Killing form, we will consider* $AV(X)$ *as a subset of* $\mathfrak{g}_{\mathbb{C}}$*.*

We will recall some results of Gyoga and Yamashita in [10]. We first introduce some additional notation. Let $\mathfrak{w}$ be a subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{w}^*$ and ${\mathfrak{g}_{\mathbb{C}}}^*$ denote their complex dual spaces, i.e., $Hom(\mathfrak{w},\ \mathbb{C})$ and $Hom(\mathfrak{g}_{\mathbb{C}},\ \mathbb{C})$. Let $p : {\mathfrak{g}_{\mathbb{C}}}^* \to \mathfrak{w}^*$ denote the projection mapping.

Definition 4.3. *(GY-condition) An element* $\lambda \in {\mathfrak{g}_{\mathbb{C}}}^*$ *satisfies condition* $P_{\mathfrak{k}_{\mathbb{C}},\ \mathfrak{w}}$*, if* $p(\mathfrak{k}_{\mathbb{C}} \cdot \lambda) = \mathfrak{w}^*$*. Here,* $\mathfrak{k}_{\mathbb{C}} \cdot \lambda = \{z \cdot \lambda | z \in \mathfrak{k}_{\mathbb{C}}\}$ *and "$\cdot$" denotes the coadjoint action.*

A subalgebra $\mathfrak{w}$ for which there is a $\lambda \in {\mathfrak{g}_{\mathbb{C}}}^*$ satisfying condition $P_{\mathfrak{k}_{\mathbb{C}},\ \mathfrak{w}}$ in Definition 4.3 is said to satisfy the Gyoja-Yamashita condition (abbreviated GY-condition). Gyoja and Yamashita establish the following result.

Theorem 4.1. *(Theorem 2.1 in* [10]*) Let* X *be an irreducible* $(\mathfrak{g}_{\mathbb{C}},\ K)$*-module, and* $\mathfrak{w}$ *be a subalgebra of* $\mathfrak{g}_{\mathbb{C}}$*. Suppose that there is an element* $\lambda \in AV(X)$ *which satisfies condition* $P_{\mathfrak{k}_{\mathbb{C}},\ \mathfrak{w}}$ *in Definition 4.3. Then the action of the enveloping algebra* $U(\mathfrak{w})$ *on* X *is locally free, i.e.,* X *is a torsion free* $U(\mathfrak{w})$*-module.*

Definition 4.4. *Recall that* $B_{\mathfrak{g}_{\mathbb{C}}}$ *denotes the Killing form of* $\mathfrak{g}_{\mathbb{C}}$*. Let* τ *denote the conjugation of* $\mathfrak{g}_{\mathbb{C}}$ *with respect to the compact real form* $\widetilde{u} = \mathfrak{k} + i\mathfrak{p}$*. Let* $\widetilde{U}$ *be the connected subgroup of* $G_{\mathbb{C}}$ *with Lie algebra* $\widetilde{u}$*. Then,*

$$\langle z,\ w\rangle := -B_{\mathfrak{g}_{\mathbb{C}}}(z,\ \tau(w)) \tag{13}$$

is a U*-invariant positive definite Hermitian form on* $\mathfrak{g}_{\mathbb{C}}$*.*

One can verify that $\langle \mathfrak{k}_{\mathbb{C}}(x;j),\ \mathfrak{p}_{\mathbb{C}}(x;n)\rangle = 0$, for all integers j and n, and $\langle (\mathfrak{p}_{\mathbb{C}}(x;j),\ \mathfrak{p}_{\mathbb{C}}(x : n)\rangle$ vanishes identically if and only if $j \neq n$. The form $\langle \cdot,\ \cdot\rangle$ in equation (13) allows us to identify $\mathfrak{p}_{\mathbb{C}}(x;i)$ with $\mathfrak{p}_{\mathbb{C}}(x;i)^*$ for each i. We will establish a slight variant of Theorem 4.1 for a situation in which $\mathfrak{w}$ is only assumed to be a subspace. By the observation in Remark 2.1, the vector space $\widetilde{V}$ satisfies the GY-condition using the element e. The key fact is that: $[\mathfrak{q}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}},\ e] = \widetilde{V}$. This can be converted into the appropriate statement concerning duals.

Lemma 4.1. *Suppose that* X *is an irreducible Harish Chandra module, and* v_0 *is a non zero vector in* X*. Set* M_0 *equal to a finite dimensional* K*-stable subspace of* X *which contains* v_0*. Form the* $S(\mathfrak{g}_{\mathbb{C}})$*-module* $gr(X) := gr(X; M_0)$*. (See equation (12).) Denote by* $\widetilde{v_0}$ *the vector in* $M_0 \subset gr(X)$ *that corresponds to* v_0 *in* X*. Assume that* W *is a vector subspace of* $\mathfrak{g}_{\mathbb{C}}$ *that satisfies the* GY*-condition for one of its elements* w'*. Then:*

(a)$Ann_{S(W)}\widetilde{v_0} = (0)$*, i.e., no non-zero element of* $S(W)$ *annihilates* $\widetilde{v_0}$*.*

(b) No non-zero element of $\sigma(S(W))$ *annihilates* v_0*, where* σ *is the symmetrization map from* $S(\mathfrak{g}_{\mathbb{C}})$ *onto* $U(\mathfrak{g}_{\mathbb{C}})$*.*

Proof. Statement(a) of the lemma is established as in Proposition 2.1 of [10]. The fact that W is not necessarily a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ does not affect the proof.

Now suppose that $g = \sum_{i=0}^{n} g_i \in S(W)$, where for each i, $g_i \in S^i(W)$. Then $\sigma(g) = \sum \sigma(g_i)$. If $0 = \sigma(g)v_0 = \sum \sigma(g_i)v_0$, then, $\sigma(g_n)v_0 = -\sum_{i=0}^{n-1} \sigma(g_i)v_0$. Assume that for at least one $i < n$, $\sigma(g_i) \neq 0$. (Otherwise, $\sigma(g_n)v_0 = 0$. In which case, since $g_n\widetilde{v_0} = \sigma(g_n)v_0 + U_{n-1}M_0$, $g_n\widetilde{v_0} = 0$.) Then $\sigma(g_n)v_0 \in U_{n-1}v_0$ which implies that $g_n\widetilde{v_0} = 0$ but then $g_n = 0$ by part (a). In the same way, we prove that $g_{n-1},\ g_{n-2}, \dots,\ g_2,\ g_1,\ g_0$ are each equal to zero. □

Proposition 4.1. *Suppose that X is an irreducible $(\mathfrak{g}_{\mathbb{C}},\ K)$-module, and $AV(X) = \overline{\mathcal{O}}$, where $\mathcal{O}$ is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$ that is small and spherical. With notation as above, for each $i = 1, \dots r$, u_i acts injectively on X.*

Proof. This is a consequence of the preceding lemma and the definition of the u_i above in (11). □

Lemma 4.2. *With notation as above, for all i and j, $u_iu_j - u_ju_i \in \sigma(S(\mathfrak{p}_{\mathbb{C}}))U(\mathfrak{n}_k)$.*

Proof. For each i, $u_i = \sigma(f_i)$. Either all f_i belong to $S(\mathfrak{p}_{\mathbb{C}}(x;2))$ or all f_i belong to $S(w_0(\mathfrak{p}_{\mathbb{C}}(x;-2))$. We may assume that x was chosen to be dominant relative to $\mathfrak{n}_k$. The assertion of the lemma follows from applying several facts about commutation. First,

$$[\mathfrak{p}_{\mathbb{C}}(x;2),\ \mathfrak{p}_{\mathbb{C}}(x;2)] \subset \mathfrak{k}_{\mathbb{C}}(x;4) \subset \mathfrak{n}_k, \text{ and } [\mathfrak{p}_{\mathbb{C}}(x;2),\ \mathfrak{k}_{\mathbb{C}}(x;4)] = 0.$$

Similarly,

$$[w_0\mathfrak{p}_{\mathbb{C}}(x;-2),\ w_0\mathfrak{p}_{\mathbb{C}}(x;-2)] \subset w_0\mathfrak{k}_{\mathbb{C}}(x;-4) \subset \mathfrak{n}_k,$$

and

$$[w_0\mathfrak{p}_{\mathbb{C}}(x;-2),\ w_0\mathfrak{k}_{\mathbb{C}}(x;-4)] = 0.$$

□

Remark 4.1. *The previous lemma along with Proposition 4.1 implies that if $w \in X^{\mathfrak{n}_k}$, and $w \neq 0$, then $u_1^{m_1}u_2^{m_2}\dots u_r^{m_r} \cdot w$ is independent of the order in which the $u_1^{m_1}, \dots,\ u_r^{m_r}$ are applied to w.*

Corollary 4.1. *Assume the hypotheses of Proposition 4.1. Let μ be a K-type in X and let X^{μ} denote the corresponding highest weight space, then for each i, multiplication by $\sigma(f_i)$ maps X^{μ} to $X^{\mu+\gamma_i}$ injectively. Thus, for any choice of non-negative integers $m_1, \dots, m_r$, $\mu + \sum_{i=1}^{r} m_1\gamma_i$ is a K-type of X.*

In [2] Barbasch and Vogan define a notion of asymptotic K-support for an irreducible unitary representation of G. Let us extend their definition to completely reducible K modules whose K-multiplicities are suitably bounded. Assume that $\mathcal{R}$ is a completely reducible K-module with finite K-type multiplicities. Let $\mathcal{R}(\lambda)$ denote the λ-isotypic component for $\lambda \in \widehat{K}$, and V_λ denote an irreducible finite dimensional K-module with highest weight λ. If $m(\lambda) = m_{\mathcal{R}}(\lambda)$ is the multiplicity of λ in $\mathcal{R}$, we will assume that there is some constant κ (independent of λ) such that $m(\lambda) \leq \kappa \dim V_\lambda$ for all $\lambda \in \widehat{K}$.

Definition 4.5. *(See Theorem 3.6 in* [2]*)*

$$\begin{aligned}&AS^K(\mathcal{R}) = \{\delta \in \mathfrak{t}^* | \exists \textit{ a sequence } \delta_n \in \widehat{K},\ n \in \mathbf{N},\ \textit{with } \mathcal{R}(\delta_n) \neq 0, \forall n,\\ &\textit{and a sequence of positive numbers}, t_n, \textit{with } \lim_{n\to\infty} t_n = 0,\\ &\textit{such that } \lim_{n\to\infty} t_n\delta_n = \delta\}\end{aligned}$$

We call $AS^K(\mathcal{R})$ the set of asymptotic directions of the K-types of $\mathcal{R}$. $\mathbf{R}^+\left(\Gamma(\overline{\mathcal{O}})\right)$ is clearly the set of asymptotic directions of the K-types in $R[\overline{\mathcal{O}}]$.

Theorem 4.2. *Suppose that X is an irreducible $(\mathfrak{g}_{\mathbb{C}},\ K)$-module, and $AV(X) = \overline{\mathcal{O}}$, where $\mathcal{O}$ is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$ that is small and spherical. Then,*

$$AS^K(X) = \mathbf{R}^+\left(\Gamma(\overline{\mathcal{O}})\right),$$

i.e., the set of asymptotic directions of the K-types of X is the same as the set of asymptotic directions of the K-types of $R[\overline{\mathcal{O}}]$.

Proof. The proof depends on the following facts: for any choice of a finite dimensional , K-stable subspace M_0 in equation (12), (1) X and $gr(X)$ have the same K multiplicities, and (2) $gr(X)$ is a finitely generated $(R[\overline{\mathcal{O}}],\ K)$ module. See Chapter 11 of [25].

We first show that $AS^K(R[\overline{\mathcal{O}}]) \subset AS^K(gr(X))$. Suppose $\delta \in AS^K(R[\overline{\mathcal{O}}])$, and $\mathfrak{t}_n$, and δ_n are as in Definition 4.5. Let μ be a K-type of X, then by Corollary 4.1, $\mu + \delta_n$ is a K-type of X. Then $\lim_{n\to\infty} t_n(\mu + \delta_n) = \delta$. So $\delta \in AS^K(gr(X))$.

Second, we show that $AS^K(gr(X)) \subset AS^K(R[\overline{\mathcal{O}}])$. Since $gr(X)$ is a finitely generated $(R[\overline{\mathcal{O}}],\ K)$ module, we can find V, a finite dimensional K submodule of $gr(X)$, such that the map $R[\overline{\mathcal{O}}] \otimes V \to gr(X)$, defined by $\sum(m_i \otimes v_i) \mapsto \sum m_i v_i$, is a surjective K homomorphism. Let $y_1, \ldots,\ y_j$ be a basis of $\mathfrak{t}$-weight vectors of V and let μ_i be the weight of y_i. Now assume that $\delta \in AS^K(gr(X))$. Choose $\mathfrak{t}_n$, and δ_n as in Definition 4.5. Then each δ_n has the form $\mu_{i_n} + {\delta_n}'$ where ${\delta_n}'$ is a K-type of $R[\overline{\mathcal{O}}]$. (This follows from the fact that if V_λ is an irreducible K-submodule of $R[\overline{\mathcal{O}}]$, each irreducible submodule W of $V_\lambda \otimes V$ has highest weight $\lambda + \mu_i'$ for some weight μ_i' of V.) But $||\mu_{i_n}||$ is bounded as $n \to \infty$, since the μ_{i_n} come from a finite set. It follows that $\delta = \lim_{n\to\infty} t_n\delta_n = \lim_{n\to\infty} t_n{\delta_n}'$. This implies that $\delta \in AS^K(R[\overline{\mathcal{O}}])$. □

Example 4.1. *Set $\mathfrak{g} = sl(4,\ \mathbf{R})$ and $\mathfrak{k} = so(4)$. We consider the K-type decomposition of the Speh representation of $SL(4,\ \mathbf{R})$, with Knapp-Vogan parameter $m = -2$, and lowest K type $(1,\ 1)$. We denote this representation by $S[(-2,\ (1,1)]$, and refer the reader to pages 586-588 of* [14] *for further discussion of its properties. Let $Y_1,\ Y_2,\ Y_3$ be the following complex symmetric matrices:*

$$\begin{bmatrix} 1 & -i & 0 & 0\\ -i & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -i\\ 0 & 0 & -i & -1\end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -i\\ 0 & 0 & -i & -1\\ 1 & -i & 0 & 0\\ -i & -1 & 0 & 0\end{bmatrix}.$$

Let $\mathcal{O}$ be the $K_{\mathbb{C}}$ orbit of $e = Y_1 + Y_2$. $\mathcal{O}$ is a spherical nilpotent of height 2.

The K-decomposition of $S[(-2,\ (1,1)]$ is as follows:

$$S[(-2,\ (1,1)] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} V(2m+1,\ 2n+1),$$

where $V(2m+1,\ 2n+1)$ is the irr. rep. of $\mathfrak{k}_{\mathbb{C}}$ with highest weight $(2m+1,\ 2n+1)$.

The associated variety of $S[(-2,\ (1,1)]$ is $\overline{\mathcal{O}}$. Each of the Y_i ($i = 1,2,3$) defined above can be viewed as a function on $\mathfrak{p}_{\mathbb{C}}$, via $Y_i(Z) = trace(Y_i Z)$. With this identification, the following functions generate $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$: Y_1 (weight $(2,0)$) and $Y_1Y_2 - (1/4)Y_3^2$ (weight $(2,2)$). The corresponding elements of $U(\mathfrak{g}_{\mathbb{C}})$ under the symmetrisation map act injectively on $S[-2,\ (1,1)]$. The highest weights of the K-types of $S[-2,\ (1,1)]$ may be obtained by successive application of these elements to the highest weight of $(1,\ 1)$, the lowest K-type of $S[-2,\ (1,1)]$.

5. The coordinate rings of small spherical classes of the exceptional simple algebras

In this section we will give the degrees and highest weights of the generators of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ for each small non-trivial spherical nilpotent orbit $\mathcal{O}$ of an exceptional simple Lie algebra. For the simple algebras other than EIV and GI, this information is contained in the tables in sub-section 5.13. If $\mathfrak{g} = EIV$ or GI, then there is only one non-trivial spherical nilpotent orbit, and these cases are discussed separately below.

We use the following conventions. For each exceptional simple Lie algebra, we fix a positive system for $\mathfrak{g}_{\mathbb{C}}$ (relative to a fundamental Cartan subalgebra of $\mathfrak{g}$) as in Appendix C of [13]. (The corresponding set of simple roots is the "Vogan" diagram of $\mathfrak{g}$.) Each nilpotent class in $\mathfrak{p}_{\mathbb{C}}$ is described by a weighted Dynkin diagram relative to a set of simple roots for $\mathfrak{k}_{\mathbb{C}}$ as in [6] and [11]. A simple root of $\mathfrak{g}_{\mathbb{C}}$ is denoted by α_i and the corresponding fundamental weight is denoted by ν_i. Since $\mathfrak{g}$ is simple and exceptional, $[\mathfrak{k}_{\mathbb{C}},\ \mathfrak{k}_{\mathbb{C}}]$ is either (1) simple, or (2) has the form $\mathfrak{a} \oplus sl(2,\ \mathbb{C})$ where $\mathfrak{a}$ is simple. In case (1), a simple root of $[\mathfrak{k}_{\mathbb{C}},\ \mathfrak{k}_{\mathbb{C}}]$ is denoted by β_j and the corresponding fundamental weight is denoted by ω_j. In case (2) a simple root of $\mathfrak{a}$ is denoted by β_j and the corresponding fundamental weight is denoted by ω_j. In case (2), $\widetilde{\omega_1}$ denotes the fundamental representation of the $sl(2,\ \mathbb{C})$ summand. $B = B_{\mathfrak{g}_{\mathbb{C}}}$ is the Killing form of $\mathfrak{g}_{\mathbb{C}}$ which is used to identify $\mathfrak{g}_{\mathbb{C}}$ with its complex dual.

Here is an explanation of the column labels of the tables below.

"Class" refers to the number of the nilpotent orbit as it is listed in the tables in [6] and [11]. "Class i" for a particular $\mathfrak{g}$ will sometimes be denoted by '$\mathcal{O}_i$'.

"$x(\mathcal{O})$" is the weighted Dynkin diagram of the orbit $\mathcal{O}$, relative to a set of simple roots for $\mathfrak{k}_{\mathbb{C}}$ chosen as in the tables in [6] and [11]. That is, $x(\mathcal{O})$ gives the values of the $\beta_j(x)$ for the neutral element $x = x(\mathcal{O})$ corresponding to $\mathcal{O}$. Often $x(\mathcal{O})$ is omitted because of space considerations.

"$\dim \mathcal{O}$" is the complex dimension of the orbit $\mathcal{O}$.

"$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$" lists the simple summands in $(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss} = [\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}},\ \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}]$, where $\mathfrak{l}_{\mathbb{C}} = {\mathfrak{g}_{\mathbb{C}}}^x$ as above.

"$\mathfrak{p}_{\mathbb{C}}(x;2)$" gives the structure of $\mathfrak{p}_{\mathbb{C}}(x;2)$ as an $(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$-module. This information is copied from and is explained fully in [11]. For example, for class 6 in the table for $EVII$, E_6 signifies one of the 26 dimensional irreducible representations of the Lie algebra E_6. When $(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$ is not simple, it is usually clear from the entry in column "$\mathfrak{p}_{\mathbb{C}}(x;2)$" which summands of $(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$ are acting non-trivially. For example, for class 2 in the table for EI, the A_1 summand of $(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$ acts trivially on $\mathfrak{p}_{\mathbb{C}}(x;2)$, and the B_2 summand acts by its first fundamental representation.

"$(d_i,\ \gamma_i)$" lists the degree and the highest weight of the generators of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$. This information is obtained by first constructing the roots of $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$ and weight vectors in $\mathfrak{p}_{\mathbb{C}}(x;2)$ and examining $S[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)}$ as we did in Example 3.2. Information about the weights of the generators $S(\mathfrak{p}_{\mathbb{C}}(x;2)^{u(\mathfrak{l}_k)}$ is obtained from [4] and [16]. Few details are given.

"Action of h_ν": When $\mathfrak{k}_{\mathbb{C}}$ is not semisimple, i.e., in case $\mathfrak{g} = EIII$ or $EVII$ the center of $\mathfrak{k}_{\mathbb{C}}$ is spanned by an element h_ν in the Cartan subalgebra which corresponds to a particular fundamental weight of $\mathfrak{g}_{\mathbb{C}}$. This column gives the action of h_ν on each generator of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$.

5.1. *EI*. $\mathfrak{k} = sp(4)$. See Table 1.

Classes 1 and 2 have height 2. For classes 1 and 2, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2. For class 1, the generator of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ is the highest weight of the representation of K on $\mathfrak{p}_{\mathbb{C}}$ (i.e., ω_4, the fourth fundamental representation of $sp(4)$.) $\mathfrak{p}_{\mathbb{C}}(x;2)$ is a one dimensional $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$-module, and so has $[\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}},\ \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}]$-weight equal to zero.

For class 2, from [11], we know the structure of $\mathfrak{p}_{\mathbb{C}}(x;2)$ as an $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$ module. $[\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}},\ \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}]$ is isomorphic to $so(5,\ \mathbb{C})$ and $\mathfrak{p}_{\mathbb{C}}(x;2)$ is λ_1, the first fundamental representation of $so(5,\ \mathbb{C})$. Clearly $\lambda_1 = \omega_4$. Looking at $x(\mathcal{O})$, we see that $[\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}},\ \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}]$ corresponds to the simple roots β_4 and β_3 for $\mathfrak{k}_{\mathbb{C}}$. Denote the highest weight vector in $\mathfrak{p}_{\mathbb{C}}(x;2)$ by v_1. From [4], $S[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)}$ has two generators; one in degree 1 of weight λ_1 and the second in degree 2 of weight zero which is an invariant for $so(5,\ \mathbf{C})$. Hence $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ has a generator in degree 1 and a generator in degree 2. The generator in degree 1 can be taken to be the function $B(v_1,\ \cdot)$ which has weight ω_4. The generator in degree 2 can be determined as follows. Assume that w_1 is a lowest weight vector in $\mathfrak{p}_{\mathbb{C}}(x;2)$, and v_1 is as before. Then as an element in $S^2(\mathfrak{p}_{\mathbb{C}}(x;2))$, the second degree generator has the form $\sum_i v_i \otimes w_i + w_i \otimes v_i$, with $v_i,\ w_i \in \mathfrak{p}_{\mathbb{C}}(x;2)$. Let $w_0(B_2)$ denote the longest element of the Weyl group generated by the simple reflections corresponding to the simple roots β_3 and β_4. Then $w_0(B_2)\omega_4$ is the lowest weight of $\mathfrak{p}_{\mathbb{C}}(x;2)$. It follows that the second degree generator has weight:

$$\omega_4 + w_0(B_2)\omega_4 = 2\omega_4 - 2\beta_3 - 2\beta_4 = 2\beta_1 + 4\beta_2 + 4\beta_3 + 2\beta_4 = 2\omega_2.$$

5.2. *EII*. $\mathfrak{k} = su(6) \oplus su(2)$. See Table 2.

Classes 1, 2, and 3 have height 2. For class 1, the generator of $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ is the highest weight of the representation of K on $\mathfrak{p}_{\mathbb{C}}$, i.e., $\omega_3 \otimes \widetilde{\omega_1}$ (the outer tensor product of the third fundamental representation of $su(6)$ and the fundamental representation of $su(2)$.

For class 2, the label $SO(6)$ for $\mathfrak{p}_{\mathbb{C}}(x;2)$ indicates the 6 dimensional representation of A_3 on $\wedge^2\mathbb{C}^4$, i.e., λ_2, the second fundamental representation of A_3. In this case, $\lambda_2 = \omega_3 \otimes \widetilde{\omega_1}$, and corresponds to a generator in degree 1. The second generator lies in degree 2, and is determined as we did for class 2 for EI. That is, we compute: $\lambda_2 + w_0(A_3)\lambda_2 = (\omega_1 + \omega_5) \otimes \widetilde{\omega_1}$. Both $\omega_3 \otimes \widetilde{\omega_1}$ and $(\omega_1 + \omega_5) \otimes \widetilde{\omega_1}$ are self dual.

For class 3, $(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss} = 4su(2)$, where two of $su(2)$ summands act trivially on $\mathfrak{p}_{\mathbb{C}}(x;2)$ and the sum of the other two summands can be identified with $so(4,\ \mathbb{C})$. Thus, the $(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$-module $S(\mathfrak{p}_{\mathbb{C}}(x;2)$ can be viewed as the $so(4,\ \mathbb{C})$-module $S(\mathbb{C}^4)$ where $\mathbb{C}^4$ is the fundamental representation λ_1 of $so(4,\ \mathbb{C})$. The highest weights of the generators of the highest weights of the $so(4)$-module $S(\mathbf{C}^4)$ are λ_1 and 0. Since $\lambda_1 = \omega_3 \otimes \widetilde{\omega_1}$ and $0 = \lambda_1 + w_0(2A_1)\lambda_1$, the highest weights for class 3 are $\omega_3 \otimes \widetilde{\omega_1}$ and $(\omega_2 + \omega_4) \otimes \widetilde{\omega_1}$. Both highest weights are self dual.

For classes 1, 2 and 3, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2.

5.3. *EIII***.** $\mathfrak{k} = so(10) \oplus so(2)$. See Table 3.

Class 1 is the orbit of the root vector $X_{-\alpha_6}$, where α_6 is the unique non compact simple root in the Vogan diagram for $\mathfrak{g}$. $R[\overline{\mathcal{O}_1}]^{\mathfrak{n}_k}$ is generated by the linear function $B(X_\gamma,\ \cdot)$ where X_γ is the root vector for γ, the longest root of $\mathfrak{g}$. This function has weight ω_4 for $so(10)$, since $\gamma = \omega_4 + (3/4)\nu_6$. Let h_{ν_6} be the element of $\mathfrak{t}_{\mathbb{C}}$ that corresponds to the fundamental weight ν_6 under the Killing form. The center of $\mathfrak{k}_{\mathbb{C}}$ is spanned by the vector h_{ν_6} which acts by $(\alpha_6,\ \alpha_6)/2$ on the irreducible representation with highest weight vector γ. (The inner product is defined by the Killing form.) Class 2 is the orbit of the root vector X_γ. $R[\overline{\mathcal{O}_2}]^{\mathfrak{n}_k}$ is generated by the linear function $B(X_{-\alpha_6}, \cdot)$. One checks that $-\alpha_6 = \omega_5 - (3/4)\nu_6$, and so corresponds to the weight ω_5 of $so(10)$. $h_{\widetilde{\lambda_6}}$ acts by $-(\alpha_6,\ \alpha_6)/2$ on the irreducible representation with highest weight vector $B(X_{-\alpha_6},\ \cdot)$. It is also clear that $\alpha_6 = w_0(D_5)\gamma$.

For class 3, $S(\mathfrak{p}_{\mathbb{C}}(x;2)^{u(\mathfrak{l}_k)}$ has two generators, in degree 1 and degree 2 respectively. Their weights are determined as for class 2 in *EI*. After applying $-w_0$ to this set of weights, one obtains $\gamma = \omega_4 + (3/4)\nu_6$ in degree 1, and $\omega_1 + (3/2)\nu_6$ in degree 2. Thus, h_{ν_6} acts on the second degree generator by $(\alpha_6,\ \alpha_6)$. The arguments for classes 4, 6 and 9 are similar. The highest degree generators of $S(\mathfrak{p}_{\mathbb{C}}(x;2)^{u(\mathfrak{l}_k)}$ occur in degree 2. Class 9 is notable since $S(\mathfrak{p}_{\mathbb{C}}(x;2)^{u(\mathfrak{l}_k)}$ has 5 generators, the most occurring for a small spherical orbit of an exceptional Lie algebra. Notice that $R[\overline{\mathcal{O}}]$ is self dual for classes 5, 6 and 9.

Classes 1–5 have height 2. For each of the classes 1–6, the boundary of $\mathcal{O}_i$ has codimension exceeding 2 in $\overline{\mathcal{O}_i}$. See [7]. Therefore, for each of these orbits, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2.

5.4. *EIV***.** $\mathfrak{k} = F_4$. Class 1 has height 2 and $R[\overline{\mathcal{O}}]$ is self dual. Class 1 is the orbit of a highest weight vector v of $\mathfrak{k}_{\mathbb{C}}$ acting on $\mathfrak{p}_{\mathbb{C}}$. $R[\overline{\mathcal{O}}]^{\mathfrak{n}_k}$ is generated by the linear function $B(v,\ \cdot)$. This function has weight ω_4, the fourth fundamental weight for $\mathfrak{k} = F_4$.

5.5. *EV***.** $\mathfrak{k} = su(8)$. See Table 4.

Classes 1–4 have height 2 and are the only small spherical orbits. $R[\overline{\mathcal{O}_1}]$ is self dual. The respective dimensions of classes 1–4 are 17, 26, 27, 27. For classes 1 and 2, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2.

5.6. *EVI***.** $\mathfrak{k} = so(12) \oplus su(2)$. See Table 5.

Classes 1–3 have height 2 and are the only small spherical classes. All spherical orbit closures have the property that $R[\overline{\mathcal{O}}]$ is self dual. This case is similar to *EII*, and our results are consistent with those of Gross and Wallach [9] who also consider class 4. For classes 1–3, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2.

Notes on Table 5: $\widetilde{\omega_1}$ denotes the fundamental representation of $su(2)$; ω_5 is one of the spin representations of $so(12)$; *triv* is the trivial representation of $su(2)$.

5.7. *EVII***.** $\mathfrak{k} = E_6 \oplus so(2)$. See Table 6.

Classes 1–9 have height 2. Class 10 is the only other small spherical one.

Class 1 is the orbit of the root vector $X_{-\alpha_7}$, where α_7 is the unique non compact simple root in the Vogan diagram for $\mathfrak{g}$. The $R[\overline{\mathcal{O}_1}]^{\mathfrak{n}_k}$ is generated by the linear function $B(X_\gamma,\ \cdot)$ where X_γ is the root vector for γ, the longest root of $\mathfrak{g}$. This function has weight ω_6 for E_6, since $\gamma = (1/3)(2\beta_1+3\beta_2+4\beta_3+6\beta_4+5\beta_5+4\beta_4)+(2/3)\nu_7$ which equals $\omega_6+(2/3)\nu_7$. Let h_{ν_7} be the element of $\mathfrak{t}_{\mathbb{C}}$ that corresponds to the fundamental weight ν_7 under the Killing form. The center of $\mathfrak{k}_{\mathbb{C}}$ is spanned by the vector h_{ν_7} which acts by $(\alpha_7,\ \alpha_7)/2$ on $B(X_\gamma,\ \cdot)$.

Class 2 is the orbit of the root vector X_γ. $R[\overline{\mathcal{O}_2}]^{\mathfrak{n}_k}$ is generated by the linear function $B(X_{-\alpha_7},\ \cdot)$. One checks that $-\alpha_7 = (1/3)(4\beta_1+3\beta_2+5\beta_3+6\beta_4+4\beta_5+2\beta_6)-(2/3)\nu_7$ which equals $\omega_1-(2/3)\nu_7$ so that the linear function $B(X_{-\alpha_7},\ \cdot)$ has weight ω_1 and h_{ν_7} acts by $-(\alpha_7,\ \alpha_7)/2$ on $B(X_{-\alpha_7},\ \cdot)$.

For classes 1–5, and 10, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2.

5.8. *EVIII*. $\mathfrak{k} = so(16)$. See Table 7.

Classes 1 and 2 have height 2. These are the only two small spherical classes. $R[\overline{\mathcal{O}}]$ is self dual for all nilpotent orbits.

Class 1 is the orbit of the root vector $X_{-\alpha_1}$, where α_1 is the unique non compact simple root in the Vogan diagram for $\mathfrak{g}$. The $R[\overline{\mathcal{O}_1}]^{\mathfrak{n}_k}$ is generated by the linear function $B(X_{-\alpha_1},\ \cdot)$ where $X_{-\alpha_1}$ is the root vector for $-\alpha_1$. Since, $-\alpha_1 = \omega_7$, this linear function has weight ω_7 for $so(16)$. ω_7 is one of the spin representations of $so(16)$.

For class 2, there is one generator of $S[\mathfrak{p}_{\mathbb{C}}(x;2)]^{u(\mathfrak{l}_k)}$ in degree 1, and a second generator in degree 2. The second degree generator is an invariant for $so(8)$. The highest weight of the representation of $\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{k}_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}(x;2)$ is $-\alpha_1$ and the lowest weight is $-\alpha_1-\beta_5-2\beta_6-2\beta_7-\beta_8$. Adding the two weights together (to get the $so(8)$ invariant gives $\beta_1+2\beta+3\beta_3+4\beta_5+4\beta_5+4\beta_6+2\beta_7+2\beta_8=\omega_4$, since $-\alpha_1=\omega_7=(1/2)(\beta_1+2\beta_2+3\beta_3+4\beta_4+5\beta_5+6\beta_6+4\beta_7+3\beta_8)$.

For classes 1 and 2, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2.

5.9. *EIX*. $\mathfrak{k} = E_7 \oplus su(2)$. See Table 8.

Classes 1–3 have height 2. This case is similar to *EII*, and the results are consistent with those of Gross and Wallach [9]. For classes 1–3, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2.

5.10. *FI*. $\mathfrak{k} = sp(3) \oplus su(2)$. See Table 9.

Classes 1–3 have height two and are the only small spherical orbits. This case is similar to *EII*, and our results are consistent with Gross and Wallach [9]. For classes 1–3, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2. For each nilpotent orbit of $\mathfrak{g}$, $R[\overline{\mathcal{O}}]$ is self dual.

5.11. *FII*. $\mathfrak{k} = so(9)$. See Table 10.

Only class 1 has height 2. For classes 1 and 2, $R[\mathcal{O}] = R[\overline{\mathcal{O}}]$ by Proposition 2.2. For each nilpotent orbit of $\mathfrak{g}$, $R[\overline{\mathcal{O}}]$ is self dual.

5.12. *GI*. $\mathfrak{k} = su(2) \oplus su(2)$. The only non-zero small spherical nilpotent orbit is class 1. Class 1 has height 2. α_2 is the unique non-compact simple root in the Vogan diagram and is long. α_1 is the compact simple root and is short. We choose the positive system for $\mathfrak{k}_{\mathbb{C}}$ given by the roots α_1 and $\psi = 3\alpha_1 + 2\alpha_2$. $R[\overline{\mathcal{O}}_1]$ is generated by a linear function with weight $3\omega_1 \otimes \widetilde{\omega_1}$ where $3\omega_1$ is the four dimensional representation of the $su(2)$ summand of $\mathfrak{k}$ corresponding to the root α_1 and $\widetilde{\omega_1}$ is the fundamental representation (i.e., the two dimensional irreducible representation) of the $su(2)$ summand of $\mathfrak{k}$ corresponding to the root ψ.

5.13. **Tables 1-10.** The notation of the following tables is explained at the beginning of section 5.

Table 1. Small spherical classes of EI

Class	$x(\mathcal{O})$	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss})$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^{r}$
1	$0-0-0 \Leftarrow 1$	11	A_3	$\mathbb{C}$	$(1,\ \omega_4)$
2	$0-1-0 \Leftarrow 0$	16	$A_1 \oplus B_2$	$SO(5)$	$(1,\ \omega_4)$ $(2,\ 2\omega_2)$

Table 2. Small spherical classes of EII

Class	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^{r}$
1	11	$A_2 \oplus A_2$	$\mathbb{C}$	$(1,\ \omega_3 \otimes \widetilde{\omega_1})$
2	16	A_3	$SO(6)$	$(1,\ \omega_3 \otimes \widetilde{\omega_1})$ $(2,\ (\omega_1+\omega_5) \otimes\ 2\widetilde{\omega_1})$
3	16	$4A_1$	$SL_2^1 \otimes SL_2^2$	$(1,\ \omega_3 \otimes \widetilde{\omega_1})$ $(2,\ (\omega_2+\omega_4) \otimes\ triv)$

Table 3. Small spherical classes of $EIII$

$Class$	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^{r}$	Action of h_{ν_6}
1	11	A_4	$\mathbb{C}$	$(1,\ \omega_4)$	$(\alpha_6,\ \alpha_6)/2$
2	11	A_4	$\mathbb{C}$	$(1,\ \omega_5)$	$-(\alpha_6,\ \alpha_6)/2$
3	16	D_4	$SO(8)$	$(1,\ \omega_4)$	$(\alpha_6,\ \alpha_6)/2$
				$(2,\ \omega_1)$	$(\alpha_6,\ \alpha_6)$
4	16	D_4	$SO(8)$	$(1,\ \omega_5)$	$-(\alpha_6,\ \alpha_6)/2$
				$(2,\ \omega_1)$	$-(\alpha_6,\ \alpha_6)$
5	16	A_3	$\mathbb{C} \oplus \mathbb{C}$	$(1,\ \omega_4)$	$(\alpha_6,\ \alpha_6)/2$
				$(1,\ \omega_5)$	$-(\alpha_6,\ \alpha_6)/2$
6	21	A_3	$SL_4 \oplus SL_4^*$	$(1,\ \omega_4)$	$(\alpha_6,\ \alpha_6)/2$
				$(1,\ \omega_5)$	$-(\alpha_6,\ \alpha_6)/2$
				$(2,\ \omega_2)$	0
9	24	D_4	$S0(8) \oplus Spin(8)$	$(1,\ \omega_4)$	$(\alpha_6,\ \alpha_6)/2$
				$(1,\ \omega_5)$	$-(\alpha_6,\ \alpha_6)/2$
				$(2,\ \omega_1)$	$(\alpha_6,\ \alpha_6)$
				$(2,\ \omega_1)$	$-(\alpha_6,\ \alpha_6)$
				$(2,\ \omega_2)$	0

Table 4. Small spherical classes of EV

$Class$	$x(\mathcal{O})$	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^{r}$
1	$0-0-0-1-0-0-0$	17	$2A_3$	$\mathbb{C}$	$(1,\ \omega_4)$
2	$0-1-0-0-0-1-0$	26	$2A_1 \oplus A_3$	$\wedge^2 SL_4$	$(1,\ \omega_4)$
					$(2,\ \omega_2+\omega_6)$
3	$0-2-0-0-0-0-0$	27	$A_1 \oplus A_5$	$\wedge^2 SL_6$	$(1,\ \omega_4)$
					$(2,\ \omega_2+\omega_6)$
					$(3,\ 2\omega_6)$
4	$0-0-0-0-0-2-0$	27	$A_5 \oplus A_1$	$\wedge^2 SL_6$	$(1,\ \omega_4)$
					$(2,\ \omega_2+\omega_6)$
					$(3,\ 2\omega_2)$

Table 5. Small spherical classes of EVI

Class	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^r$
1	17	$A_4 \oplus A_1$	$\mathbb{C}$	$(1,\ \omega_5 \otimes \widetilde{\omega_1})$
2	26	$A_1 \oplus D_4$	$SO(8)$	$(1,\ \omega_5 \otimes \widetilde{\omega_1})$
				$(2,\ \omega_2 \otimes\ 2\widetilde{\omega_1})$
3	26	$A_3 \oplus 3A_1$	$SL_2^1 \otimes SL_2^2$	$(1,\ \omega_5 \otimes \widetilde{\omega_1})$
				$(2,\ \omega_4 \otimes\ triv)$

Table 6. Small spherical classes of $EVII$

Class	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^r$	Action of h_{ν_7}
1	17	D_5	$\mathbb{C}$	$(1,\ \omega_6)$	$(\alpha_7,\ \alpha_7)/2$
2	17	D_5	$\mathbb{C}$	$(1,\ \omega_1)$	$-(\alpha_7,\ \alpha_7)/2$
3	26	D_5	$SO(10)$	$(1,\ \omega_6)$	$(\alpha_7,\ \alpha_7)/2$
				$(2,\ \omega_1)$	$(\alpha_7,\ \alpha_7)$
4	26	D_5	$SO(10)$	$(1,\ \omega_1)$	$-(\alpha_7,\ \alpha_7)/2$
				$(2,\ \omega_6)$	$-(\alpha_7,\ \alpha_7)$
5	26	D_4	$\mathbb{C} \oplus \mathbb{C}$	$(1,\ \omega_1)$	$-(\alpha_7,\ \alpha_7)/2$
				$(1,\ \omega_6)$	$(\alpha_7,\ \alpha_7)/2$
6	27	E_6	E_6	$(1,\ \omega_6)$	$(\alpha_7,\ \alpha_7)/2$
				$(2,\ \omega_1)$	$(\alpha_7,\ \alpha_7)$
				$(3,\ 0)$	$3(\alpha_7,\ \alpha_7)/2$
7	27	E_6	E_6	$(1,\ \omega_1)$	$-(\alpha_7,\ \alpha_7)/2$
				$(2,\ \omega_6)$	$-(\alpha_7,\ \alpha_7)$
				$(3,\ 0)$	$-3(\alpha_7,\ \alpha_7)/2$
8	27	D_5	$\mathbb{C} \oplus SO(10)$	$(1,\ \omega_1)$	$-(\alpha_7,\ \alpha_7)/2$
				$(1,\ \omega_6)$	$(\alpha_7,\ \alpha_7)/2$
				$(2,\ \omega_1)$	$(\alpha_7,\ \alpha_7)$
9	27	D_5	$SO(10) \oplus \mathbb{C}$	$(1,\ \omega_6)$	$(\alpha_7,\ \alpha_7)/2$
				$(1,\ \omega_1)$	$-(\alpha_7,\ \alpha_7)/2$
				$(2,\ \omega_6)$	$-(\alpha_7,\ \alpha_7)$
10	33	A_5	$SL_6 \oplus SL_6^*$	$(1,\ \omega_1)$	$-(\alpha_7,\ \alpha_7)/2$
				$(1,\ \omega_6)$	$(\alpha_7,\ \alpha_7)/2$
				$(2,\ \omega_2)$	0

Table 7. Small spherical classes of $EVIII$

Class	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^{r}$
1	29	A_7	$\mathbb{C}$	$(1,\ \omega_7)$
2	46	$A_3 \oplus D_4$	$SO(8)$	$(1,\ \omega_7)$ $(2,\ \omega_4)$

Table 8. Small spherical classes of EIX

Class	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^{r}$
1	29	E_6	$\mathbb{C}$	$(1,\ \omega_7 \otimes \widetilde{\omega_1})$
2	46	D_6	$SO(12)$	$(1,\ \omega_7 \otimes \widetilde{\omega_1})$ $(2,\ \omega_1 \otimes\ 2\widetilde{\omega_1})$
3	46	$D_5 \oplus 2A_1$	$SL_2^1 \otimes SL_2^2$	$(1,\ \omega_7 \otimes \widetilde{\omega_1})$ $(2,\ \omega_6 \otimes triv)$

Table 9. Small spherical classes of FI

Class	$x(\mathcal{O})$	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^{r}$
1	$0-0 \Leftarrow 1\ \ 1$	8	A_2	$\mathbb{C}$	$(1,\ \omega_3 \otimes \widetilde{\omega_1})$
2	$1-0 \Leftarrow 0\ \ 2$	11	B_2	$SO(5)$	$(1,\ \omega_3 \otimes \widetilde{\omega_1})$ $(2,\ 2\omega_1 \otimes 2\widetilde{\omega_1})$
3	$0-1 \Leftarrow 0\ \ 0$	11	$3A_1$	$SL_2^2 \otimes SL_2^3$	$(1,\ \omega_3 \otimes \widetilde{\omega_1})$ $(2,\ 2\omega_2 \otimes\ triv)$

Table 10. Small spherical classes of FII

Class	$x(\mathcal{O})$	$\dim \mathcal{O}$	$(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})_{ss}$	$\mathfrak{p}_{\mathbb{C}}(x;2)$	$\{(d_i,\ \gamma_i)\}_{i=1}^{r}$
1	$0-0-0 \Rightarrow 1$	11	A_3	$\mathbb{C}$	$(1,\ \omega_4)$
2	$4-0-0 \Rightarrow 0$	15	B_3	$Spin(7)$	$(1,\ \omega_4)$ $(2, \omega_1)$

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