COMPLEXITY OF NILPOTENT ORBITS AND THE KOSTANT-SEKIGUCHI CORRESPONDENCE

DONALD R. KING

Abstract. Let G be a connected linear semisimple Lie group with Lie algebra $\mathfrak{g}$, and let $K_{\mathbf{C}} \rightarrow Aut(\mathfrak{p}_{\mathbf{C}})$ be the complexified isotropy representation at the identity coset of the corresponding symmetric space. Suppose that Ω is a nilpotent G-orbit in $\mathfrak{g}$ and $\mathcal{O}$ is the nilpotent $K_{\mathbf{C}}$-orbit in $\mathfrak{p}_{\mathbf{C}}$ associated to Ω by the Kostant-Sekiguchi correspondence. We show that the corank of the Hamiltonian K-space Ω is twice the complexity of the $K_{\mathbf{C}}$ variety $\mathcal{O}$. Furthermore, the corank of Ω is a lower bound on a numerical measure of the noncommutativity of the Poisson algebra of K-invariant real analytic functions on Ω.

1. Introduction

The Kostant-Sekiguchi correspondence is a vital tool in the study of infinite dimensional representations of semisimple Lie groups. Let us recall some facts about this correspondence in case G is a connected real, linear semisimple Lie group with maximal compact subgroup K. We obtain the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of G (resp., K). (θ is the corresponding Cartan involution.) The vector spaces $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ are then complexified to give a vector space decomposition of $\mathfrak{g}_{\mathbf{C}}$, the Lie algebra of $G_{\mathbf{C}}$ (the complexification of G), as $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}$. The Kostant-Sekiguchi correspondence is a bijection between the nilpotent G orbits in $\mathfrak{g}$ and the nilpotent $K_{\mathbf{C}}$ orbits in $\mathfrak{p}_{\mathbf{C}}$. (For the precise definition of the correspondence we refer the reader to [2].)

Definition 1.1. *If Ω is a nilpotent G-orbit in $\mathfrak{g}$ and $\mathcal{O}$ is the nilpotent $K_{\mathbf{C}}$-orbit in $\mathfrak{p}_{\mathbf{C}}$ associated to Ω by the Kostant-Sekiguchi correspondence, then $(\Omega,\ \mathcal{O})$ is said to be a Kostant-Sekiguchi pair.*

Among the nice elementary properties of such a pair are: (1) Ω and $\mathcal{O}$ lie in the same $G_{\mathbf{C}}$ orbit which we denote by $\mathcal{O}_{\mathbf{C}}$ and (2) $\mathcal{O}$ is a Lagrangian submanifold of $\mathcal{O}_{\mathbf{C}}$ (relative to the Kostant-Souriau symplectic form on $\mathcal{O}_{\mathbf{C}}$.) Moreover, Vergne [10] has established a much deeper relationship between Ω and $\mathcal{O}$, namely that there is a K-equivariant diffeomorphism which maps Ω onto $\mathcal{O}$. This result is a major tool in our paper.

Recently, the author proved that if $(\Omega,\ \mathcal{O})$ form a Kostant-Sekiguchi pair then Ω is multiplicity free as a Hamiltonian K-space if and only if $\mathcal{O}$ is a spherical $K_{\mathbf{C}}$ variety [2]. (The definition of multiplicity free is given below in Remark 2.1. Spherical $K_{\mathbf{C}}$-varieties are defined below in Remark 2.4.) The goal of this paper is to prove a generalization of that result to all Kostant-Sekiguchi pairs. That generalization is contained in Theorem 3.1. It states that the corank of the Hamiltonian K-space Ω is twice the complexity of the $K_{\mathbf{C}}$ variety $\mathcal{O}$. This relationship was partially

2000 *Mathematics Subject Classification.* Primary 22E46; Secondary 14R20, 53D20.

suggested by the results of [1]. One consequence of our main result is that twice the complexity of the $K_{\mathbf{C}}$ action on $\mathcal{O}$ is a lower bound for a numerical measure (Definition 4.2) of the failure of the Poisson algebra of K-invariant real analytic functions on Ω to be commutative (Proposition 4.4).

The author wishes to thank A. T. Huckleberry for his proof of the density result in Proposition 2.1 and Maxim Braverman for some useful discussions.

2. Notation and key definitions

We now introduce some further concepts and notations. Unless otherwise indicated, in this section K will denote an arbitrary compact group. We assume that K is contained in its complexification $K_{\mathbf{C}}$. Our basic reference for symplectic manifolds with Hamiltonian K-actions is [9].

If $A \subset G_{\mathbf{C}}$ is a Lie subgroup and $S \subset \mathfrak{g}_{\mathbf{C}}$ then A^S denotes the subgroup of A that fixes each element of S under the adjoint action of $G_{\mathbf{C}}$ on $\mathfrak{g}_{\mathbf{C}}$. If $\mathfrak{a} \subset \mathfrak{g}_{\mathbf{C}}$ is a Lie subalgebra, then $\mathfrak{a}^S$ is defined similarly.

For the remainder of this section $\mathbf{X}$ will denote a connected symplectic manifold which is a Hamiltonian K-space. Let the K-invariant symplectic form be $w_{\mathbf{X}}$ and the moment map be $\Phi : \mathbf{X} \to \mathfrak{k}^*$.

Definition 2.1. *Let $\mathfrak{A} = \mathfrak{A}(\mathbf{X}) = C^\infty(\mathbf{X})^K$ be the algebra (with respect to Poisson bracket) of K invariant smooth functions on $\mathbf{X}$. $\mathfrak{Z} = \mathfrak{Z}(\mathbf{X})$ denotes the center of $\mathfrak{A}$. Let $\mathfrak{B} = \mathfrak{B}(\mathbf{X}) = \{f \in C^\infty(\mathbf{X}) | f = h \circ \Phi,$ for some $h \in C^\infty(\mathfrak{k}^*)\}$ be the algebra of collective functions on X.*

Remark 2.1. $\mathbf{X}$ *is said to be multiplicity free if $\mathfrak{A} = \mathfrak{Z}$ i.e., $\mathfrak{A}$ is a commutative Poisson algebra.*

Our main example of a symplectic manifold with Hamiltonian K-action will be a nilpotent G-orbit Ω in $\mathfrak{g}$ where G is connected semisimple Lie group and K is a maximal compact subgroup of G. We first recall some facts about the symplectic structure of Ω and the action of K on Ω. Suppose that $\Omega = G \cdot E$. For each $E' \in \Omega$, we identify $T_{E'}(\Omega)$, the tangent space of Ω at E', with the quotient $\mathfrak{g}/\mathfrak{g}^{E'}$. The Kostant-Souriau form w_Ω on Ω is defined by setting

$$w_\Omega|_{E'}(\bar{Y},\ \bar{Z}) = \kappa(E',\ [Y,\ Z]),$$

for all $Y,\ Z \in \mathfrak{g}$ where κ denotes the Killing form of $\mathfrak{g}$. ($\bar{Y}$ and $\bar{Z}$ are the cosets of Y and Z in $\mathfrak{g}/\mathfrak{g}^{E'}$.) w_Ω is a symplectic form on Ω.

The action of K on Ω is the left action. If $\xi \in \mathfrak{k}$, then ξ determines a global vector field ξ_Ω on Ω according to the following definition:

$$\xi_\Omega f(E') = \left.\frac{df(exp(-t\xi) \cdot E')}{dt}\right|_{t=0}$$

where f is any smooth function on Ω and $E' \in \Omega$. If Y is any smooth vector field on Ω, $Y(E')$ is the vector in $T_{E'}(\Omega)$ obtained by evaluating Y at E'.

Since w_Ω is invariant under G, it is invariant under K. The K action on Ω is Hamiltonian in the following sense. For each $\xi \in \mathfrak{k}$, there is a function $\phi^\xi \in C^\infty(\Omega)$ such that $\iota(\xi_\Omega)w_\Omega = -d\phi^\xi$, where $\iota(\xi_\Omega)$ denotes interior multiplication by ξ_Ω. We may take $\phi^\xi = \kappa(\xi,\ \cdot)$. We obtain the moment mapping $\Phi_\Omega : \Omega \to \mathfrak{k}^*$ by setting $\Phi_\Omega(E')(\xi) = \phi^\xi(E') = \kappa(E',\ \xi) = \kappa(E'_k,\ \xi)$ for all $E' \in \Omega$ and $\xi \in \mathfrak{k}$. E'_k denotes the component of E' in $\mathfrak{k}$. We summarize this discussion in the following remark.

Remark 2.2. *Ω is a Hamiltonian K-space, with moment mapping $\Phi_\Omega : \Omega \to \mathfrak{k}$ defined by sending an element $E' \in \Omega$ to its component in $\mathfrak{k}$. That is:*

$$\Phi_\Omega : E' \mapsto \frac{E' - \theta(E)}{2}. \tag{1}$$

(We have identified $\mathfrak{k}$ with its real dual space $\mathfrak{k}^*$ using the restriction of κ to $\mathfrak{k}$.) Each function f in $C^\infty(\Omega)$ gives rise to a smooth vector field X_f satisfying $df(Y) = w_\Omega(Y,\ X_f)$ for all smooth vector fields Y on Ω. X_f is said to be the Hamiltonian vector field associated to f. (If $\xi \in \mathfrak{k}$, then $X_{\phi^\xi} = \xi_\Omega$.) $C^\infty(\Omega)$ is a Poisson algebra under the Poisson bracket $\{\cdot,\ \cdot\}$ defined as follows: for $f,\ g \in C^\infty(\Omega)$, $\{f,\ g\} = w_\Omega(X_f,\ X_g)$. One checks that the linear mapping $\xi \mapsto \phi^\xi$ is a Lie algebra homomorphism $\mathfrak{k} \to C^\infty(\Omega)$ when $\{\cdot,\ \cdot\}$ is taken as the Lie bracket on $C^\infty(\Omega)$.

Lemma 2.1. *Let $E' \in \mathbf{X}$. Set $W = W_{E'} = T_{E'}(K \cdot E')$ and $W^\perp$ equal to the orthogonal complement (with respect to $\zeta = w_{\mathbf{X}}|_{E'}$) of W inside $T_{E'}(\mathbf{X})$, then we have the following orthogonal decomposition of $T_{E'}(\mathbf{X})$ into (real) symplectic vector spaces:*

$$T_{E'}(\mathbf{X}) = \frac{W}{W \cap W^\perp} \bigoplus \left((W \cap W^\perp) \oplus (W \cap W^\perp)^*\right) \bigoplus \frac{W^\perp}{W \cap W^\perp}.$$

The restriction of ζ to $(W \cap W^\perp) \oplus (W \cap W^\perp)^*$ is given by $\zeta((Y_1,\ \lambda_1),\ (Y_2,\ \lambda_2)) = \lambda_1(Y_2) - \lambda_2(Y_1)$ for all $Y_i \in (W \cap W^\perp)$, $\lambda_i \in (W \cap W^\perp)^*$. Moreover, (1) $\mathfrak{k}^{\Phi(E')}/\mathfrak{k}^{E'}$ and $W \cap W^\perp$ are isomorphic (as $\mathfrak{k}^{E'}$ modules) and (2) $\mathfrak{k}/\mathfrak{k}^{\Phi(E')}$ and $\dfrac{W}{W \cap W^\perp}$ are isomorphic (as $\mathfrak{k}^{E'}$ modules).

Proof. See Corollary 9.10 and Lemma 9.11 of [3]. These results are steps in the proof of the normal slice theorem of Guillemin, Sternberg and Marle. □

Definition 2.2. *If $E'' \in \mathbf{X}$ (or $\mathfrak{k}$), set $d(E'') = \dim K \cdot E''$. Let $d = d_{\mathbf{X}}$ be the maximum dimension of a K-orbit in $\mathbf{X}$ and let $m = m_{\mathbf{X}}$ denote the minimal codimension of a K-orbit in $\mathbf{X}$. d_Φ will denote the maximum dimension of a K orbit in $\Phi(\mathbf{X})$. We define three important subsets of $\mathbf{X}$.*

(2)

$$\mathbf{X}_d = \{E' \in \mathbf{X} | \dim K \cdot E' = d\}$$

$$\mathbf{X}_0 = \{E' \in \mathbf{X} | \exists \text{ an open set } U \subset \mathbf{X} \text{ such that } E' \in U \text{ and } d(\cdot) \text{ is constant on } U\}$$

$$\mathbf{X}_\Phi = \{E' \in \mathbf{X} | \dim K \cdot \Phi(E') = d_\Phi\} = \{E' \in \mathbf{X} | \dim \mathfrak{k}^{\Phi(E')} \text{ is minimum}\}$$

Remark 2.3. *From Proposition 27.1 in* [9] *we know that $\mathbf{X}_0$ is open and dense in $\mathbf{X}$. In addition, $\mathbf{X}_0 \subset \mathbf{X}_d$. Otherwise there is a point in $\mathbf{X}_0$ whose K orbit has dimension $d' < d$. But, then there must be an open subset of points whose K orbits have dimension d' which is impossible.*

Proposition 2.1. *$\mathbf{X}_\Phi$ is open and dense in $\mathbf{X}$.*

Proof. The fact that $\mathbf{X}_\Phi$ is open is Lemma 1 in section 3 of [7]. The following proof (unpublished) that $\mathbf{X}_\Phi$ is dense in $\mathbf{X}$ is due to A. T. Huckleberry.

It suffices to show that $\mathbf{X}_\Phi^c$, the complement of $\mathbf{X}_\Phi$, has codimension at least 2 in $\mathbf{X}$. We use induction on the dimension of K. If $\dim K = 0$, then K is finite. Then $\mathbf{X}_\Phi = \mathbf{X}$ since all K orbits in $\Phi(\mathbf{X})$ are zero dimensional. Suppose $\dim K > 0$. If

$x \in \mathbf{X}_\Phi^c$, we need to show that there is an open neighborhood of x whose intersection with $\mathbf{X}_\Phi^c$ has codimension at least 2. Let K_0 denote the identity component of K. First consider $\mathbf{X}^{K_0}$, the set of fixed points of K_0 on $\mathbf{X}$, and its intersection with $\mathbf{X}_\Phi^c$. If $K_0 \cdot x = x$, we use the slice theorem to construct an open neighborhood $U = K \times_{K^x} \Sigma$ of x. Define Σ_Φ relative to the action of $(K^x)_0$ in the obvious way. The argument for Proposition 27.3 in [9] shows that either (a) K_0 (which equals $(K^x)_0$) acts trivially on Σ, or (b) U^{K_0} has codimension ≥ 2 in U. In case (a) K acts as a finite group on U so that $U = U_\Phi$. In case (b) $U^{K_0} \cap (U_\Phi)^c$ has codimension ≥ 2 in U. This argument takes care of the points in $\mathbf{X}_\Phi^c \cap \mathbf{X}^{K_0}$. If $x \notin \mathbf{X}^{K_0}$, then the slice neighborhood $U = K \times_{K^x} \Sigma$ has the property that $\dim K^x < \dim K$. By induction $(\Sigma_\Phi)^c$ has codimension at least 2 in Σ. Thus $(U_\Phi)^c = K \times_{K^x} (\Sigma_\Phi)^c$ has codimension at least 2 in U. □

Since $\mathbf{X}_d$ and $\mathbf{X}_\Phi$ are each open and dense in $\mathbf{X}$, we have the following result.

Corollary 2.1. *$\mathbf{X}_\Phi \cap \mathbf{X}_d$ is open and dense in $\mathbf{X}$.*

We know recall the notion of a coisotropic submanifold of $\mathbf{X}$.

Definition 2.3. *The orbit $K \cdot E'$ in $\mathbf{X}$ is said to be coisotropic if $W_{E'}^\perp \subset W_{E'}$.*

By results of Guillemin and Sternberg [8], $\mathbf{X}$ is multiplicity free as a Hamiltonian K-space if and only if there is an open dense subset U of $\mathbf{X}$ such that for $E' \in U$ the orbit $K \cdot E'$ is coisotropic in $\mathbf{X}$. Therefore, it is reasonable to use the size of the quotient $\dfrac{T_{E'}(K \cdot E')^\perp}{T_{E'}(K \cdot E') \cap T_{E'}(K \cdot E')^\perp}$ for generic K orbits to measure the failure of $\mathbf{X}$ to be multiplicity free.

Proposition 2.2. *There is an open dense subset U of $\mathbf{X}$ such that for all $E' \in U$, the non-negative integer*

$$\dim \frac{T_{E'}(K \cdot E')^\perp}{T_{E'}(K \cdot E') \cap T_{E'}(K \cdot E')^\perp}, \tag{3}$$

has a constant value. We denote this value by $2\widetilde{\epsilon}_K(\mathbf{X})$. The integer $2\widetilde{\epsilon}_K(\mathbf{X})$ is said to be the corank of the action of K on $\mathbf{X}$ [14].

Proof. Choose $U = \mathbf{X}_d \cap \mathbf{X}_\Phi$. Then for all $E' \in U$, the decomposition of $T_{E'}(\mathbf{X})$ in Lemma 2.1 implies that

$$\dim \frac{T_{E'}(K \cdot E')^\perp}{T_{E'}(K \cdot E') \cap T_{E'}(K \cdot E')^\perp} = \dim \mathbf{X} - 2d + d_\Phi \tag{4}$$

Note that if there is another dense open set U' on which the dimension function in (3) is constant, then U' must have non-empty intersection with $\mathbf{X}_d \cap \mathbf{X}_\Phi$. Thus, $\widetilde{\epsilon}_K(\mathbf{X})$ is well defined. □

As a corollary of the preceding proof we have

Corollary 2.2. *Suppose that $E' \in \mathbf{X}_d \cap \mathbf{X}_\Phi$, then*

$$\dim \frac{T_{E'}(K \cdot E')^\perp}{T_{E'}(K \cdot E') \cap T_{E'}(K \cdot E')^\perp} = 2\widetilde{\epsilon}_K(\mathbf{X}).$$

We recall from [7] or [2] the definition of the rank of the action of K on X.

Definition 2.4. *The rank of the K-action on X, denoted by $r_K(X)$, is equal to rank K - rank $K^{E'}$, where $E' \in X$ and the orbit $K \cdot E'$ has maximum dimension among the K orbits in X.*

Finally, we recall from [4], the notions of rank and complexity of algebraic $K_{\mathbf{C}}$ actions.

Definition 2.5. *Suppose that Y is a variety with $K_{\mathbf{C}}$ action and B_k is a Borel subgroup of $K_{\mathbf{C}}$. The complex codimension of a generic B_k orbit is called the complexity of Y, denoted $c_{K_{\mathbf{C}}}(Y)$ or $c(Y)$ (when the reductive group $K_{\mathbf{C}}$ is understood). If U_k is the nilpotent radical of B_k and $B_k \cdot z$ is a generic B_k orbit, then the codimension of $U_k \cdot z$ in $B_k \cdot z$ is called the rank of Y. It is denoted $r_{K_{\mathbf{C}}}(Y)$.*

Remark 2.4. *$c(Y)$ is also the transcendence degree (over $\mathbf{C}$) of the B_k invariant functions in the field of rational functions (with complex coefficients) on Y. Y is spherical for $K_{\mathbf{C}}$ if and only if $c(Y) = 0$. The rank of Y is also the transcendence degree of the U_k invariants in the field of rational functions on Y.*

3. Main Theorem

Our main result is:

Theorem 3.1. *Let $(\mathcal{O},\ \Omega)$ be a Kostant-Sekiguchi pair, and $\Phi = \Phi_\Omega$ be the moment map in (1), then*
(a) $r_{K_{\mathbf{C}}}(\mathcal{O}) = r_K(\Omega)$;
(b) $c(\mathcal{O}) = \widetilde{\epsilon}_K(\Omega)$.

This result was inspired by the main result of [1].

We assume from now on that $(\Omega,\ \mathcal{O})$ is a Kostant-Sekiguchi pair. The proof of Theorem 3.1 requires two important facts about Ω and $\mathcal{O}$. The first is the existence of a K invariant diffeomorphism

$$\mathcal{V}_\Omega : \Omega \to \mathcal{O} \tag{5}$$

that was mentioned above. The second fact is the existence of a K-invariant diffeomorphism

$$\mathcal{M}_{\mathcal{O}} : \mathcal{O} \to K \times_{K^{\mathfrak{s}}} V_{\mathcal{O}}(\mathfrak{s}) \tag{6}$$

established in the proof of Proposition 5.2 of [2]. To describe the vector bundle $K \times_{K^{\mathfrak{s}}} V_{\mathcal{O}}(\mathfrak{s})$ in (6), we recall the notation of [2].

There is a Kostant-Sekiguchi $sl(2)$-triple $\{x,\ e,\ f\}$ such that $\mathcal{O} = K_{\mathbf{C}} \cdot e$. The Kostant-Sekiguchi property means that (1) $x \in i\mathfrak{k}$, $e,\ f \in \mathfrak{p}_{\mathbf{C}}$, (2) $e = \sigma(f)$, where σ is conjugation on $\mathfrak{g}_{\mathbf{C}}$ relative to the real form $\mathfrak{g}$, and (3) the following Lie bracket relations hold: $[x,\ e] = 2e$, $[x,\ f] = -2f$, and $[e,\ f] = x$. It follows that the Lie algebra $\mathbf{C}x \oplus \mathbf{C}e \oplus \mathbf{C}f$ is the complexification of a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, where $\mathfrak{s}$ is isomorphic to $sl(2,\ \mathbf{R})$. $V_{\mathcal{O}}(\mathfrak{s})$ is the quotient $[\mathfrak{k}_{\mathbf{C}},\ e]/[\mathfrak{k},\ e]$. In [2] it is shown that $K \times_{K^{\mathfrak{s}}} V_{\mathcal{O}}(\mathfrak{s})$ is diffeomorphic to the conormal bundle of $K \cdot e$ inside the cotangent bundle of $\mathcal{O}$. In addition, there is an isomorphism of $K^{\mathfrak{s}}$ modules over $\mathbf{R}$:

$$V_{\mathcal{O}}(\mathfrak{s}) \simeq \mathfrak{k}^x/\mathfrak{k}^{\mathfrak{s}} \oplus Z. \tag{7}$$

where Z is the sum (over $\mathbf{C}$) of the positive eigenspaces of $ad(x)$ on $\mathfrak{k}_{\mathbf{C}}$ that do not lie in $\mathfrak{k}^e_{\mathbf{C}}$. (In equation (7), Z is being regarded as a real vector space.)

We now establish the main result, Theorem 3.1.

Proof. We first consider part (a).

By composing the K-invariant diffeomorphisms in (5) and (6), we have the assignment: $E' \mapsto \mathcal{M}_{\mathcal{O}} \circ \mathcal{V}(E')$. So we can identify E' with an equivalence class $[k_0,\ y+z]$ in $K \times_{K^{\mathfrak{s}}} V_{\mathcal{O}}(\mathfrak{s})$ where $y \in \mathfrak{k}^x/\mathfrak{k}^{\mathfrak{s}}$ and $z \in Z$. (See equation (7).) It is more convenient to consider the equivalence class $\mathcal{M} \circ \mathcal{V}(k_0^{-1} \cdot E') = [\mathbf{1},\ y+z]$ where $\mathbf{1}$ denotes the identity in K. Assume that $\dim K \cdot E' = d_\Omega$ (Definition 2.2). Set $\mathbf{s_R} = (\mathfrak{k}^{\mathfrak{s}})^y$ and $\mathbf{s_C} = (\mathfrak{k}^{\mathfrak{s}}_{\mathbf{C}})^y$. Set $\mathbf{S}$ equal to the connected subgroup of $K^{\mathfrak{s}}_{\mathbf{C}}$ with Lie algebra $\mathbf{s_C}$ and $\mathbf{S_R}$ equal to the connected subgroup of $\mathbf{S}$ with Lie algebra $\mathbf{s_R}$. Since $K \cdot E'$ has maximum dimension among the K orbits in Ω, (1) $(\mathfrak{k}^{\mathfrak{s}})^{y+z}$ has minimum dimension among the subalgebras $(\mathfrak{k}^{\mathfrak{s}})^v$ for $v \in V_{\mathcal{O}}(\mathfrak{s})$; (2) $(\mathfrak{k}^{\mathfrak{s}})^y$ has minimum dimension among the subalgebras $(\mathfrak{k}^{\mathfrak{s}})^{y'}$ for $y' \in \mathfrak{k}^x/\mathfrak{k}^{\mathfrak{s}}$; and (3) $\dim \mathbf{S} \cdot z = max_{z' \in Z} \dim \mathbf{S} \cdot z'$. (See Lemma 6.2 of [2].)

$\mathbf{S}$ (resp., $\mathbf{S_R}$) is a stabilizer of general position for the action of $K^{\mathfrak{s}}_{\mathbf{C}}$ on $\mathfrak{k}^x_{\mathbf{C}}/\mathfrak{k}_{\mathbf{C}}{}^{\mathfrak{s}}$ (resp., $\mathfrak{k}^x/\mathfrak{k}^{\mathfrak{s}}$). (For basic facts on points in general position, and stabilizers of general position for algebraic group actions, see [4], [5] and [6].) Also, $\mathfrak{k}^{[\mathbf{1},\ y+z]}$, the centralizer of $[\mathbf{1},\ y+z]$ in $\mathfrak{k}$ is equal to $\mathbf{s_R}{}^z$ (which is the same as $(\mathfrak{k}^{\mathfrak{s}})^{y+z}$). Since $\mathcal{M}_{\mathcal{O}} \circ \mathcal{V}$ is K-equivariant, $\mathfrak{k}^{k_0{}^{-1} \cdot E'} = \mathfrak{k}^{[\mathbf{1},\ y+z]}$. Hence, since $\mathfrak{k}^{E'} = Ad(k_0)(\mathfrak{k}^{k_0{}^{-1} \cdot E'})$, $\mathfrak{k}^{E'}$ and $\mathbf{s_R}{}^z$ are isomorphic Lie algebras. Therefore,

$$r_K(\Omega) := \text{rank } \mathfrak{k}^{\Phi(E')} - \text{rank } \mathfrak{k}^{E'} = \text{rank } K - \text{rank } \mathbf{S_R}{}^z. \tag{8}$$

In order to relate $r_K(\Omega)$ to $r_{K_{\mathbf{C}}}(\mathcal{O})$, we begin with a result of Panyushev (Theorem 2.3 in [5]):

$$r_{K_{\mathbf{C}}}(\mathcal{O}) = r_{K^x_{\mathbf{C}}}(K^x_{\mathbf{C}}/K^{\mathfrak{s}}_{\mathbf{C}}) + r_{\mathbf{S}}(Z). \tag{9}$$

$r_{K^x_{\mathbf{C}}}(K^x_{\mathbf{C}}/K^{\mathfrak{s}}_{\mathbf{C}})$ and $r_{\mathbf{S}}(Z)$ can computed using some additional results of Panyushev.

If $\mathbf{L}$ is a complex reductive group acting on a quasiaffine algebraic variety Y, Panyushev has shown how to define a "dual" action of $\mathbf{L}$ on Y. Denote by Y^*, the dual action of $\mathbf{L}$ on Y. Panyushev has proven the following formula for $r_{\mathbf{L}}(Y)$.

Proposition 3.1. *(Panyushev,* [6]*) Let $\widetilde{\mathbf{L}}$ be a stabilizer of general position for the diagonal action of $\mathbf{L}$ on $Y \times Y^*$, then $\widetilde{\mathbf{L}}$ is reductive and $r_{\mathbf{L}}(Y) = rank(\mathbf{L}) - rank(\widetilde{\mathbf{L}})$.*

We can apply Proposition 3.1 to action of $K^x_{\mathbf{C}}$ on $K^x_{\mathbf{C}}/K^{\mathfrak{s}}_{\mathbf{C}}$. Since $K^{\mathfrak{s}}_{\mathbf{C}}$ is reductive, $K^x_{\mathbf{C}}/K^{\mathfrak{s}}_{\mathbf{C}}$ is affine. So by Theorem 0.1(iii) and Theorem 0.2(iii) of [6] we may assume that $\mathbf{S}$ is a subgroup in general position for the diagonal action of $K^x_{\mathbf{C}}$ on

$$K^x_{\mathbf{C}}/K^{\mathfrak{s}}_{\mathbf{C}} \times (K^x_{\mathbf{C}}/K^{\mathfrak{s}}_{\mathbf{C}})^*. \tag{10}$$

It follows that

$$r_{K^x_{\mathbf{C}}}(K^x_{\mathbf{C}}/K^{\mathfrak{s}}_{\mathbf{C}}) = \text{ rank } K^x_{\mathbf{C}} - \text{rank } \mathbf{S} = \text{ rank } K^x - \text{rank } \mathbf{S_R}. \tag{11}$$

We can apply similar reasoning to the action of $\mathbf{S}$ on Z to conclude that

$$r_{\mathbf{S}}(Z) = \text{rank } \mathbf{S} - \text{rank } \mathbf{S}^z = \text{rank } \mathbf{S_R} - \text{rank } \mathbf{S_R}{}^z. \tag{12}$$

(If $\widetilde{\mathbf{S}}$ is a subgroup in general position for the diagonal action of $\mathbf{S}$ on $Z \times Z^*$, then $\widetilde{\mathbf{S}}$ is also a subgroup in general position for the action of $\mathbf{S}$ on Z. The set of points in Z whose stabilizer in $\mathbf{S}$ is conjugate to $\widetilde{\mathbf{S}}$ contains an open dense subset $\widetilde{Z}$ of Z. So we may assume that $z \in \widetilde{Z}$, and hence that $\mathbf{S}^z$ is conjugate to $\widetilde{\mathbf{S}}$. So by

Proposition 3.1 $r_{\mathbf{S}}(Z) = \text{rank } \mathbf{S} - \text{rank } \widetilde{\mathbf{S}} = \text{rank } \mathbf{S} - \text{rank } \mathbf{S}^z$.) Hence, by equations (9), (11) and (12)

$$r_{K_{\mathbf{C}}}(\mathcal{O}) = \text{rank } K - \text{rank } {\mathbf{S}_{\mathbf{R}}}^z, \tag{13}$$

which is the same as $r_K(\Omega)$ by equation (8). This concludes the proof of part (a) of Theorem 3.1.

${\mathbf{S}_{\mathbf{R}}}^z$ is a principal isotropy subgroup ([15]) for the action of $K^{\mathfrak{s}}$ on $V_{\mathcal{O}}(\mathfrak{s})$. The proof of equation (8) can be adapted to establish the following more general result:

Corollary 3.1. *Let J be a connected compact group, M a connected compact subgroup acting on the finite dimensional real vector space W, and $\widetilde{M}$ a principal isotropy subgroup for the action of M on W. Then, the M-rank of the vector bundle $J \times_M W$ is equal to rank(J)-rank($\widetilde{M}$).*

The argument for part (b) of the theorem starts by noting that for all $E' \in \Omega$, Lemma 2.1 implies that if $W = T_{E'}(K \cdot E')$, then

$$\dim \Omega = \dim T_{E'}(\Omega) = 2 \dim \left(\mathfrak{k}^{\Phi(E')}/\mathfrak{k}^{E'}\right) + \dim \left(\mathfrak{k}/\mathfrak{k}^{\Phi(E')}\right) + \dim \frac{W^{\perp}}{W \cap W^{\perp}}. \tag{14}$$

Therefore,

$$\dim \Omega = 2(\dim \mathfrak{k}^{\Phi(E')} - \dim \mathfrak{k}^{E'}) + \dim \mathfrak{k} - \dim \mathfrak{k}^{\Phi(E')} + \dim \frac{W^{\perp}}{W \cap W^{\perp}}. \tag{15}$$

Choose $E' \in \Omega_d \cap \Omega_\Phi$. By Lemma 2 in section 3 of [7], we have $[\mathfrak{k}^{\Phi(E')},\ \mathfrak{k}^{\Phi(E')}] \subset \mathfrak{k}^{E'}$. Since $\mathfrak{k}^{E'} \subset \mathfrak{k}^{\Phi(E')}$, $[\mathfrak{k}^{\Phi(E')},\ \mathfrak{k}^{\Phi(E')}] = [\mathfrak{k}^{E'},\ \mathfrak{k}^{E'}]$.

Since $\mathfrak{k}^{\Phi(E')}$ and $\mathfrak{k}^{E'}$ have the same maximum semisimple ideal,

$$\dim \mathfrak{k}^{\Phi(E')} - \dim \mathfrak{k}^{E'} = \text{rank } \mathfrak{k}^{\Phi(E')} - \text{rank } \mathfrak{k}^{E'}. \tag{16}$$

Applying Corollary 2.2 and rearranging terms in equation (15), we obtain

$$\dim \Omega = \dim \mathfrak{k}^{\Phi(E')} - \dim \mathfrak{k}^{E'} + \dim \mathfrak{k} - \dim \mathfrak{k}^{E'} + 2\widetilde{\epsilon_K}(\Omega). \tag{17}$$

Since $E' \in \Omega_d \cap \Omega_\Phi$, equation (16) implies

$$\dim \Omega = \text{rank } \mathfrak{k}^{\Phi(E')} - \text{rank } \mathfrak{k}^{E'} + \dim \mathfrak{k} - \dim \mathfrak{k}^{E'} + 2\widetilde{\epsilon_K}(\Omega),$$

where $\dim \mathfrak{k} - \dim \mathfrak{k}^{E'}$ is the maximal dimension of a K orbit in Ω.

We have just shown that m_Ω, the minimal codimension of a K orbit in Ω, is given by the expression

$$\begin{aligned} &\text{rank } \mathfrak{k}^{\Phi(E')} - \text{rank } \mathfrak{k}^{E'} + 2\epsilon(\Omega) \\ &= r_K(\Omega) + 2\widetilde{\epsilon_K}(\Omega). \end{aligned} \tag{18}$$

On the other hand, by equation (6.8) in [2], $m_{\mathcal{O}}$, the minimal codimension of a K-orbit in $\mathcal{O}$ is given by:

$$r_K(\Omega) + 2(c_{K^x_{\mathbf{C}}}(K^x_{\mathbf{C}}/K^{\mathfrak{s}}_{\mathbf{C}}) + c_{\mathbf{S}}(Z)). \tag{19}$$

Since there is a K-equivariant diffeomorphism between Ω and $\mathcal{O}$, and $c(\mathcal{O}) = c_{K^x_{\mathbf{C}}}(K^x_{\mathbf{C}}/K^{\mathfrak{s}}_{\mathbf{C}}) + c_{\mathbf{S}}(Z)$ (Theorem 2.3 in [5]), equation (19) says that the minimal codimension of a K-orbit in Ω is also given by the expression:

$$r_K(\Omega) + 2c(\mathcal{O}). \tag{20}$$

Part (b) of the theorem follows from equating (18) and (20).

□

4. Related results on the Poisson algebra of K-invariant real analytic functions on Ω

We recall the notation of sections 2 and 3. Ω has the structure of a real analytic manifold. Let $\mathfrak{A}_{ra} = \mathfrak{A}(\Omega)_{ra}$ be the Poisson algebra of K-invariant real analytic functions on Ω, and $\mathfrak{Z}_{ra}(\Omega)$ denote its center.

Definition 4.1. *Suppose $\mathbf{M}$ is a smooth manifold and $p \in \mathbf{M}$. $f_1, \ldots, f_n \in C^\infty(\mathbf{M})$ are said to be functionally independent at p if their differentials are linearly independent at p.*

Definition 4.2. *Let $\alpha_K(\Omega)$ be the largest non negative integer α such that for some dense open set $U^\flat \subset \Omega$ there functions $f_1, \ldots, f_\alpha \in \mathfrak{A}_{ra}$, (1) $f_1, \ldots, f_\alpha$ are functionally independent at each p in $U^\flat$ and (2) no nontrivial linear combination of the f_i belongs to $\mathfrak{Z}_{ra}(\Omega)$.*

Remark 4.1. *$\alpha = \alpha_K(\Omega)$ is well defined since the intersection of any two dense open sets is open and dense. Clearly $\alpha \leq \dim \Omega$.*

We see that $\alpha_K(\Omega)$ is a rough measure of the noncommutativity of $\mathfrak{A}_{ra}$.

Proposition 4.1. *Let J be a compact group acting on a finite dimensional real vector space V. Let $m = m_{\{V,J\}}$ be the codimension of a J-orbit of maximal dimension in V. Then there exist m polynomials $q_1, \ldots, q_m$ in $\mathbf{R}[V]^J$ that are functionally independent on an open dense subset of V.*

Proof. We must apply some standard facts about the action of compact groups on finite dimensional vector spaces. First, it is well known that m is equal to the dimension of the ring $\mathbf{R}[V]^J$ of J-invariant real valued polynomials on V, and hence equal to the dimension of the orbit space V/J. $\mathbf{R}[V]^J$ is graded and finitely generated. Let $q_1, \ldots q_n$ be a finite set of homogeneous generators of $\mathbf{R}[V]^J$ and let $q = (q_1, \ldots q_n)$ be the mapping $V \to \mathbf{R}^n$. Then q induces a continuous map $\eta : V/J \to \mathbf{R}^n$ of the orbit space V/J. The maps q and η are proper maps and the image $q(V)$ is a closed semialgebraic subset of $\mathbf{R}^n$. Moreover, η maps V/J homeomorphically onto $q(V)$. It follows that $q(V)$ has dimension m. Therefore, on an open dense subset of V, the rank of the Jacobian matrix $\{\frac{\partial q_i}{\partial x_j}\}$ $(1 \leq i \leq n$, $1 \leq j \leq \dim V)$ must be m. By renumbering the q_i if necessary, we can find generators $q_1, \ldots, q_m$ so that the corresponding minor of the Jacobian matrix has a non vanishing determinant on an open subset of V. Since the q_i's are polynomials, this determinant must be non zero on an open dense subset of V. Hence, $q_1, \ldots, q_m$ are functionally independent on a open dense subset of V. □

Corollary 4.1. *Let Ω be a nilpotent G-orbit in $\mathfrak{g}$. Let $m = m_{\{\Omega,K\}}$ be the codimension of a K-orbit of maximal dimension in Ω. There exists an open dense subset $U^\sharp \subset \Omega$ and functions $f_1, \ldots, f_m$ in $\mathfrak{A}_{ra}$ which are functionally independent on $U^\sharp$.*

Proof. By Proposition 2.25 of [13] that there is a K-equivariant diffeomorphism (in fact an analytic isomorphism) between Ω and the vector bundle $K \times_{K^{\mathfrak{s}}} V_\Omega(\mathfrak{s})$:

$$\mathcal{M}_\Omega : \Omega \to K \times_{K^{\mathfrak{s}}} V_\Omega(\mathfrak{s}). \tag{21}$$

This is the analog of (6) for Ω. We refer the reader to [13] for the precise definition of the vector space $V_\Omega(\mathfrak{s})$. A key fact is that m is equal to the codimension of

the largest $K^{\mathfrak{s}}$-orbit in $V_\Omega(\mathfrak{s})$. So by Proposition 4.1, we first find m K^S-invariant polynomials, $q_1, \ldots,\ q_m$, on $V_\Omega(\mathfrak{s})$ that are functionally independent on a dense open subset of $V_{\mathcal{O}}(\mathfrak{s})$. We extend these polynomials to K-invariant real analytic functions on the vector bundle $K \times_{K^{\mathfrak{s}}} V_\Omega(\mathfrak{s})$ in the obvious way. These extensions $\widetilde{q}_1, \ldots, \widetilde{q}_m$ are functionally independent on a dense open subset of the vector bundle. The diffeomorphism $\mathcal{M}_\Omega$ in (21 allows us to pull back the $\widetilde{q}_1, \ldots, \widetilde{q}_m$ to Ω. Then, $f_i = \widetilde{q}_i \circ \mathcal{M}_\Omega^{-1}$ are the required K-invariant real analytic functions on Ω which are functionally independent on a dense open subset $U^\sharp$ of Ω. □

Since the set $U^\sharp$ is open and nonempty, Corollary 2.1 implies that $U^\sharp \cap \Omega_d \cap \Omega_d$ is nonempty.

Lemma 4.1. *Suppose that $E' \in U^\sharp \cap \Omega_d \cap \Omega_\Phi$ with the notation of Definition (2.1), and Definition (4.2).*

(a) $W = T_{E'}(K \cdot E')$ is spanned by $\{X_f(E') | f$ is a collective function on $\Omega\}$.

(b) $W^\perp$ is spanned by $\{X_f(E') | f \in \mathfrak{A}_{ra}\}$. If $M_{E'}$ denotes a complement of $W \cap W^\perp$ in $W^\perp$, then $M_{E'}$ is spanned by $\{X_f(E') | f \in \mathfrak{A}_{ra},\ f \notin \mathfrak{Z}_{ra}\}$.

Proof. (a) This follows from the fact that the K action on Ω is Hamiltonian. We can identify W with the quotient $\mathfrak{k}/\mathfrak{k}^{E'}$. Set $q_{E'} = \dim W$. The restriction of the Killing form κ to $\mathfrak{k}$ is negative definite. Let z_j be a basis of the orthogonal complement (relative to the restriction of κ to $\mathfrak{k}$) of $\mathfrak{k}^{E'}$ in $\mathfrak{k}$. Consider the functions $\phi_j = \phi^{z_j}$, $(j = 1, \ldots, q_{E'})$. These functions are clearly collective. W is spanned by the tangent vectors $X_{\phi_j}(E')$.

(b) $\{X_f(E') | f \in \mathfrak{A}_{ra}\} \subset W^\perp$ because the fact that $\{f,\ g\} = 0$ for all collective functions g ([8]) implies that $w_\Omega(X_f(E'),\ X_g(E')) = 0$, for all collective functions g. The desired containment follows from (a). In addition, suppose $h_1, \ldots, h_m$ are K-invariant real analytic functions which are functionally independent on some dense open subset $U^\sharp \subset \Omega$ as in Corollary 4.1. Since we may assume that $E' \in U$, we have that the span of the linearly independent vectors $X_{h_i}(E')$ $(i = 1, \ldots, m)$ belongs to $W^\perp$. But $\dim W^\perp$ is equal to $\dim T_{E'}(\Omega) - \dim W = m$. Therefore, $W^\perp$ is spanned by $\{X_f(E') | f \in \mathfrak{A}_{ra}\}$.

If $g \in \mathfrak{Z}_{ra}$, it is clear that $X_g(E') \in M_{E'}^\perp$, since $\{f,\ g\} = 0$ for all $f \in \mathfrak{A}$. Also clearly $X_g(E') \in W^\perp \subset (W \cap W^\perp)^\perp$. It follows that $X_g(E') \in (W^\perp)^\perp = W$. That is, $g \in \mathfrak{Z}$ implies $X_g(E') \in W \cap W^\perp$. So we conclude that $M_{E'}$ is spanned by $\{X_f(E') | f \in \mathfrak{A}_{ra},\ f \notin \mathfrak{Z}_{ra}\}$.

□

Proposition 4.2. $\alpha_K(\Omega) \geq 2\widetilde{\epsilon}_K(\Omega)$.

Proof. Keep the notation of Lemma 4.1. Let $\alpha = \alpha_K(\Omega)$, and let $f_1, \ldots, f_\alpha$ belong to $\mathfrak{A}_{ra}$ and satisfy the conditions of Definition 4.2. Recall that by the choice of E', the corank of Ω is equal to $\dim \dfrac{W^\perp}{W \cap W^\perp}$. We show that the map

$$\Theta : \sum c_i f_i \mapsto \sum c_i \overline{X_{f_i}(E')}$$

from the linear span of the f_i to $\dfrac{W^\perp}{W \cap W^\perp}$ is surjective.

Suppose that Θ is not surjective, then for some $\xi \in W^\perp$ and $\xi \notin W$, the difference

$$\xi - \sum c_i X_{f_i}(E') \notin W \cap W^\perp \tag{22}$$

for any choice of constants $c_1, \ldots, c_\alpha$. By Lemma 4.1(b) there exists $g \in \mathfrak{A}_{ra}$ and $g \notin \mathfrak{Z}_{ra}$ such that $\xi = X_g(E')$. It follows that for all constants c_i, $X_g(E') - \sum c_i X_{f_i}(E') \neq 0$. Thus, the functions $g, f_1, \ldots, f_\alpha$ are functionally independent at E'. By analytic continuation the functions are functionally independent on a dense open set of Ω. By the definition of α, some linear combination of $g, f_1, \ldots, f_\alpha$ must belong to $\mathfrak{Z}_{ra}$. Since no linear combination of $f_1, \ldots, f_\alpha$ belongs to $\mathfrak{Z}_{ra}$, we can find constants d_i $(1 \leq i \leq \alpha)$, not all zero, such that $g - \sum d_i f_i \in \mathfrak{Z}_{ra}$. It follows that $X_g(E') - \sum d_i X_{f_i}(E') \in W \cap W^\perp$. This contradicts (22). □

Thus, the corank of Ω provides a lower bound for a numerical measure of the noncommutativity of the Poisson algebra of K-invariant real analytic functions on Ω.

Remark 4.2. *The author is not aware of any orbit Ω for which $\alpha_K(\Omega) > 2\widetilde{\epsilon}_K(\Omega)$.*

Example 4.1. *Let $\mathfrak{g} = sl(3, \mathbf{R})$ ($\mathfrak{k} = so(3)$) and Ω be the 6-dimensional nilpotent orbit. Then $c(\mathcal{O}) = 1$. Can one find three K-invariant real analytic functions (f_1, f_2, f_3) whose differentials are linearly independent (on a dense open subset of Ω) such that no linear combination of the f_i belongs to $\mathfrak{Z}_{ra}$.*

Remark 4.3. $\alpha_K(\Omega) = 0 \iff \Omega$ *is multiplicity free.*

Proposition 4.3. *If $s = 2\widetilde{\epsilon}_K(\Omega)$ is the corank of the K action on Ω, there exists an open dense subset U^* of Ω, such that for some set of functions $f_1, \ldots, f_s \in \mathfrak{A}_{ra}$: (1) $f_1, \ldots, f_s$ are functionally independent on U^*; (2) no nontrivial linear combination of the f_i belongs to $\mathfrak{Z}$; and (3) at each point $p \in U^*$, $f_1, \ldots, f_s$ are functionally independent of the space of collective functions on Ω. s is the largest integer for which there exists some set of s real analytic functions $f_1, \ldots, f_s \in \mathfrak{A}$ and an open dense subset U satisfying (1), (2) and (3).*

Proof. Let $d = d_\Omega$ (the maximum dimension of a K orbit of Ω). We can find an open dense set $V \subset U^\sharp \cap \Omega_d \cap \Omega_\Phi$ and a set of real analytic collective functions $g_i = \widetilde{g}_i \circ \Phi$, $i = 1, \ldots, d$, that are functionally independent on V. Set $\alpha = \alpha_K(\Omega)$. Let $f_1, \ldots, f_\alpha$ belong to $\mathfrak{A}_{ra}$, and let Y be an open dense subset of Ω so that the conditions of Definition 4.2 are satisfied. Consider the map $D : V \cap Y \to \mathbf{R}^{d+\alpha}$

$$E \mapsto (g_1(E), \ldots, g_d(E), f_1(E), \ldots, f_\alpha(E)). \tag{23}$$

If $E \in V \cap Y$, set $W = T_E(K \cdot E)$. The tangent vectors $X_{g_i}(E)$ span W, and the tangent vectors $X_{f_i}(E)$ span $W^\perp$. Therefore, the rank of the Jacobian of D at each point in $V \cap Y$ is $\dim W + \dim W^\perp - \dim(W \cap W^\perp) = d + s$. It follows that there are at most s functions assertions (1), (2) and (3) of the proposition. Moreover, it is clear that we can find s functions among the f_i and a dense open subset U^* inside $V \cap Y$ so that assertions (1), (2) and (3) are satisfied. □

Combining Proposition 4.3 and part (a) of Theorem 3.1, we obtain.

Proposition 4.4. *Let $(\mathcal{O}, \Omega)$ be a Kostant-Sekiguchi pair, then $2c_{K_{\mathbf{C}}}(\mathcal{O})$ is the largest integer n for which there exists some set of n real analytic functions $f_1, \ldots, f_n$ in $\mathfrak{A}$ and a dense open subset of Ω satisfying (1), (2) and (3) of Proposition 4.3.*

5. An Application to the G-saturation of the nilradical of a parabolic subalgebra of $\mathfrak{g}$

The folowing proposition shows that in certain cases the complexity and rank of $\mathcal{O}$ can be computed in terms of the complexity and rank of affine homogeneous spaces.

Proposition 5.1. *Let $(\Omega,\ \mathcal{O})$ form a Kostant-Sekiguchi pair. Let $P \subset G$ be a parabolic subgoup. Let N be the unipotent radical of P and set $M = P \cap K$. Let $\mathfrak{p}$, $\mathfrak{m}$, and $\mathfrak{n}$ denote the Lie algebras of P, M, and N respectively. Assume that Ω is an open subset of $G \cdot \mathfrak{n}$ (equivalently Ω has maximal dimension among the G orbits in $G \cdot \mathfrak{n}$). Let $\mathfrak{m}^{\perp}$ be an orthogonal complement to $\mathfrak{m}$ in $\mathfrak{k}$ for some K-invariant inner product on $\mathfrak{k}$ and $\widetilde{M}$ be a principal isotropy group for the action of M on $\mathfrak{m}^{\perp}$. Then,*

(a) $c_{K_{\mathbf{C}}}(\mathcal{O}) = c_{K_{\mathbf{C}}}(K_{\mathbf{C}}/M_{\mathbf{C}})$ and

(b) $r_{K_{\mathbf{C}}}(\mathcal{O}) = \ rank(K) - rank(\widetilde{M})$.

Proof. We need some facts about $T^*(G/P)$ and the G-equivariant map

$$\rho : T^*(G/P) \to G \cdot \mathfrak{n}$$

given by $(gP,\ Y) \mapsto g \cdot Y$.

There are a finite number of G-orbits in $G \cdot \mathfrak{n}$, hence a finite number of orbits of maximal dimension $= 2 \dim \mathfrak{n}$. If Ω' is an orbit of maximal dimension, it must be the complement in $G \cdot \mathfrak{n}$ of the union of the closures of the other orbits. Hence Ω' must be open in $G \cdot \mathfrak{n}$. Let $(G \cdot \mathfrak{n})_{reg}$ (resp. $T^*(G/P)_{reg}$) denote the union of the set of orbits of maximum dimension in $G \cdot \mathfrak{n}$ (resp. $T^*(G/P)$). Since $(G \cdot \mathfrak{n})_{reg}$ is open (being the union of open sets) and ρ is continuous we must have $T^*(G\ /\ P)_{reg} = \rho^{-1}(G \cdot n_{reg})$ is an open subset, hence open submanifold of $T^*(G\ /\ P)$. So we can consider the map $\rho : T^*(G/P)_{reg} \to (G \cdot \mathfrak{n})_{reg}$. If $Y \in \mathfrak{n} \cap \Omega$ where $\Omega \subset (G \cdot \mathfrak{n})_{reg}$, we can show that $d\rho_{[(eP,\ Y)]}$ is an isomorphism between tangent spaces. Therefore, by the inverse function theorem there are open subsets $U \subset T^*(G/P)_{reg}$ and $V \subset (G \cdot \mathfrak{n})_{reg}$, such that $[(eP,\ Y)] \in U$ and $Y \in V$, and $\rho : U \to V$ is a diffeomorphism. Then $\rho\ :\ U \cap \rho^{-1}(\Omega)\ \to\ V \cap \Omega$ is a G-equivariant diffeomorphism, in fact an analytic isomorphism, which preserves the restriction of the symplectic form of $T^*(G/P)$.(See Lemme 18 in [11].) It follows that $\epsilon_K(T^*(G\ /\ P) = \epsilon_K(\Omega)$. But $\epsilon_K(T^*(G/P) = \epsilon_K(T^*(K/M)$ which, by Proposition 1 and Theorem 17 in [1], is equal to $c_{K_{\mathbf{C}}}(K_{\mathbf{C}}/M_{\mathbf{C}})$. This establishes (a).

To prove (b), we first observe that by Theorem 3.1, $r_{K_{\mathbf{C}}}(\mathcal{O}) = r_K(\Omega)$. $r_K(\Omega) = r_K(T^*(G/P)) = r_K(T^*(K/M))$ because $\rho^{-1}(\Omega)$ is open in $T^*(G/P)$, and there is a K-invariant diffeomorphism $G/P \to K/M$. Let $\mathfrak{m}^{\perp}$ be an orthogonal complement to $\mathfrak{m}$ in $\mathfrak{k}$ for some K-invariant inner product on $\mathfrak{k}$. Then $T^*(K/M) = K/M \times_M \mathfrak{m}^{\perp}$. By Corollary 3.1, $r_K(T^*(K/M)) = \ \mathrm{rank}(K) - \mathrm{rank}(\widetilde{M})$. □

Remark 5.1. *Part (b) of the previous proposition is closely related to Theorem 2 in chapter 2 of* [14].

Remark 5.2. *We first recall some additional properties of complexity and rank. If Z is a $K_{\mathbf{C}}$ variety and $R[Z]$ denotes the ring of regular functions on Z, if $\lambda \in \widehat{K}$ (the set of equivalence classes of irreducible representations of K), $R[Z]_{\lambda}$ is the λ-isotypic component of $R[Z]$. $m_Z(\lambda)$ is the multiplicity of λ in $R[Z]$.*

$\Gamma(Z) := \{\lambda \in \widehat{K} | R[Z]_\lambda \neq (0)\}$ is a finitely generated semigroup in the set of dominant weights of $K_{\mathbf{C}}$. $r_{K_{\mathbf{C}}}(Z)$ is equal to the rank of the semigroup $\Gamma(Z)$.

By Panyushev [4], *$c_{K_{\mathbf{C}}}(K_{\mathbf{C}}/M_{\mathbf{C}})$ is the smallest non-negative integer c such that for all $\lambda \in \Gamma(K_{\mathbf{C}}/M_{\mathbf{C}})$:*

$$m_{K_{\mathbf{C}}/M_{\mathbf{C}}}(n\lambda) \text{ grows no faster than } n^c \text{ as } n \to \infty$$

The integer $c_{K_{\mathbf{C}}}(K_{\mathbf{C}}/M_{\mathbf{C}})$ can be calculated by a result of Heckman (Theorem 3.9 in [12] *with $\mu = 0$). In Heckman's notation,*

$$c_{K_{\mathbf{C}}}(K_{\mathbf{C}}/M_{\mathbf{C}}) = card(R_K^+) \ - \ card(R_M^+) \ - \ rank(A).$$

We recall the notation from [12] *used in the previous equation. Choose a maximal torus $\mathfrak{t}_M$ inside $\mathfrak{t}$, the maximal torus of $\mathfrak{k}$. Let $q : \mathfrak{t} \to \mathfrak{t}_M$ be the projection map. Let R_M and R_K be the root systems for M and K respectively. Set R_H equal to the subset of roots ϕ in R_K such that $q(\phi) = 0$. Choose $H_o \in \mathfrak{t}_M$ such that $\phi(H_o) \neq 0$ for all $\phi \in R_K \setminus R_H$. Fix a positive system R_H^+ for the root system R_H. Then $R_M^+ = \{\phi \in R_M | \phi(H_o) > 0\}$ is a positive system for R_M and $R_K^+ = R_H^+ \cup \{\phi \in R_K \setminus R_H | \phi(H_o) > 0\}$ is a positive system for R_K. A denotes the set $q(R_K^+ \setminus R_H^+) \setminus R_M^+$ with multiplicity.*

Example 5.1. *$\mathfrak{g} = sl(n, \mathbf{R})$. It is well known that for each nilpotent orbit Ω there is a parabolic subalgebra with nilradical $\mathfrak{n}$ such that Ω is open in $G \cdot \mathfrak{n}$. Therefore the complexity and rank of the corresponding $K_{\mathbf{C}}$-orbit $\mathcal{O}$ can be computed by Proposition 5.1.*

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