Gröbner Bases and Standard Monomial Bases

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Abstract. Let G be a semisimple algebraic group, nd P a maximal parabolic subgroup. In this note we construct in a characteristic-free way Gröbner bases for the homogeneous ideals of G/P and its Schubert varietes (for the canonical projective embedding). As a consequence, we obtain Gröbner bases for the ideals of varietes appearing in classical invariant theory, and also varietes of complexes.

Version française abrégéé. Gröbner bases give an effective way of dealing with problems concerning ideals of polynomial rings (such as the ideal membership problem, solving polynomial equations, elimination theory and computing syzygies). The idea behind Gröbner bases is to reduce a problem about a given ideal of a polynomial ring to a problem about a monomial ideal, namely the ideal generated by the leading monomials of the elements in the original ideal; using Gröbner bases, one can solve the problem in an algorithmic fashion.

In [SW] reduced Gröbner bases for the homogeneous ideals defining Grassmannians have been constructed using the "sraightening algorithm", and the authors pose the problem of extending the connection between the "sraightening algorithm" and Gröbner bases to arbitrary matroids, and in particular to Schubert varietes. Also, in [CC] reduced Gröbner bases for the homogeneous ideals of G/P, for a minuscule P, have been constructed,in characteristic 0, and the problem of pushing the correspondence to the non-minuscule case has been posed. In this note we provide answers to these problems in a characteristic-free way for G/P and its Schubert varietes, G being semisimple and P a maximal parabolic subgroup.

Let G be a semisimple, simply connected algebric group over an algebraically closed field of arbitrary characteristic. Let T be a maximal torus, B a Borel subgroup, $B \supset T$ and P be a maximal parabolic subgroup, $P \supset B$. Let W (resp. W_P) be the Weyl group of G (resp. P). For $w \in W/W_P$, let $X(w) = \overline{BwP}(\text{mod } P)$ be the Schubert variety corresponding to w. Let L be the ample generator of $\text{Pic}(G/P)$. For the projective embeding (cf. [LMS], $[\text{LS}]_2$) $X(w) \hookrightarrow G/P \hookrightarrow \text{Proj}(H^0(G/P, L))$, let R (resp. $R(w)$) denote the homogeneous coordinate ring of G/P (resp. $X(w)$). Let I (resp. $I(w)$) be the corresponding homogeneous ideal. In $[\text{LS}]_2$ (see also [LMS]), a basis for R (resp. $R(w)$) has been constructed in terms of "standard monomials". Using the "straightening relations" (cf. [LMS]), we construct reduced Gröbner bases for I and $I(w)$. As expected from the results in [SW] and [CC], the reduced Gröbner bases consist of the degree 2 straightening relations, giving the expression of a degree 2 nonstandard monomial as a linear combination of standard monomials.

§1. Gröbner Bases

1.1 Let k be an algebraically closed field of arbitrary characteristic. Consider the ring $k[x_1, x_2 \ldots, x_n]$ of polynomials in n variables $x_1, x_2 \ldots, x_n$, totally ordered as $x_1 \succ x_2 \succ \ldots \succ x_n$. A monomial of degree r will be written as $x_{i_1} x_{i_2} \ldots x_{i_r}$, with $i_1 \geq i_2 \geq \ldots \geq i_r$. Then the total order on the variables is extended to a total order on monomials in $k[x_1, x_2 \ldots, x_n]$, called the *reverse lexicographic order*, as follows: $x_{i_1} x_{i_2} \ldots x_{i_r} \prec_{rlex} x_{j_1} x_{j_2} \ldots x_{j_s}$ if and only if either $r < s$, or $r = s$ and there exists an $l < r$ such that $i_1 = j_1,\ i_2 = j_2,\ \ldots,\ i_l = j_l,\ i_{l+1} > j_{l+1}$. If f is a nonzero polynomial in $k[x_1, x_2 \ldots, x_n]$, then the highest monomial (with respect to the above total ordering) occuring in f is called the *leading monomial of* f, and we denote it by lead(f); the coefficient of lead(f) is called the leading coefficient of f. For a family of polynomials $\mathcal{F} \subset k[x_1, x_2 \ldots, x_n]$, the ideal generated by its elements will be denoted by $\langle \mathcal{F} \rangle$ and the set of the leading monomials of all polynomials in $\mathcal{F}$ will be denoted by lead$(\mathcal{F})$.

Definition 1.2 Let $I \subset k[x_1, x_2 \ldots, x_n]$ be an ideal. A family of polynomials $\mathcal{F} \subset I$ is called a *Gröbner basis for* I if $\langle \mathrm{lead}(\mathcal{F}) \rangle = \langle \mathrm{lead}(I) \rangle$.

Definition 1.3 A *minimal Gröbner basis for* I is a Gröbner basis $\mathcal{F}$ for I such that the leading coefficients of the elements in $\mathcal{F}$ are all 1 and for any $f \in \mathcal{F}$, $\mathrm{lead}(f) \notin \langle \mathrm{lead}(\mathcal{F} \setminus \{f\}) \rangle$.

Definition 1.4 A*reduced Gröbner basis for* I is a Gröbner basis $\mathcal{F}$ for I such that the leading coefficients of the elements in $\mathcal{F}$ are all 1 and for any $f \in \mathcal{F}$, no monomial present in f lies in $\langle \mathrm{lead}(\mathcal{F} \setminus \{f\}) \rangle$.

Remark 1.5 Any Gröbner basis for I generates I as an ideal.

Remark 1.6 In the case when I is the defining ideal of an algebraic variety X, a Gröbner basis for I will be also called a *Gröbner basis for* X.

Proposition 1.7 (cf. [CLO]) A nonzero ideal $I \subset k[x_1, x_2 \ldots, x_n]$ has a unique reduced Gröbner basis.

§2. First Fundamental Theorem

Let G, B, T, P, L, W, W_P, $X(w)$, etc. as above. In the sequel we work with W^P, the set of minimal representatives of W/W_P and denote the Schubert varietes in G/P by $\{X(w), w \in W^P\}$. Let R be the root sytem of G relative to T, and R^+ the system of positive roots of R relative to B. Let ω be the fundamental weight corresponding to P. For simplicity of exposition, we shall suppose P is of *classical type* (cf. [LMS], [LS]$_2$), i.e. $(\omega, \alpha^*) \leq 2$, $\alpha \in R^+$. We recall the Bruhat order in W^P, namely $w_1 \geq w_2 \iff X(w_1) \supseteq X(w_2)$. Let $X(w_2)$ be a Schubert divisor of $X(w_1)$, where $w_1 = w_2 s_\alpha$ for some $\alpha \in R^+$. We define the multiplicity of the divisor $X(w_2)$ in $X(w_1)$ to be the integer $m(w_2, w_1) = (\omega, \alpha^*)$.

Definition 2.1 A pair of elements $\underline{\tau} = (\tau^{(1)}, \tau^{(2)})$ in W^P is called an admissible pair if either $\tau^{(1)} = \tau^{(2)}$, or $\tau^{(1)} \neq \tau^{(2)}$ and there exists $\{\phi_i\}$, $1 \leq i \leq s$, $\phi_i \in W^P$ such that

(i) $\tau^{(1)} = \phi_1 > \phi_2 > \ldots > \phi_s = \tau^{(2)}$

(ii) $X(\phi_i)$ is of codimension one in $X(\phi_{i-1})$, and $m(\phi_i, \phi_{i-1}) = 2$.

We recall the following result (cf. LMS], [LS]$_2$):

Theorem 2.2 (First Basis Theorem). There exists a basis $\{p_{\underline{\tau}}\}$ for $H^0(G/P, L)$, indexed by the admissible pairs $\underline{\tau} = (\tau^{(1)}, \tau^{(2)})$ in W^P, such that

(i) $p_{\underline{\tau}}$ is a weight vector of weight $-\frac{1}{2}(\tau^{(1)}(\omega) + \tau^{(2)}(\omega))$

(ii) the restriction of $p_{\underline{\tau}}$ to a Schubert variety $X(w)$ is not identically zero if and only if $w \geq \tau^{(1)}$

(iii) the linear system $H^0(G/P,L)$ on G/P gives an embedding $G/P \hookrightarrow \mathrm{Proj}(H^0(G/P,L))$.

As above, let us denote the homogeneous coordinate ring of G/P by R.

2.3 Monomial order. We define a total order on the set $\{\underline{\tau} = (\tau^{(1)}, \tau^{(2)})\}$ of all admissible pairs in W^P as follows: we first define a partial order, namely,

$$\underline{\tau}_1 \succeq \underline{\tau}_2 \Longleftrightarrow \text{either } \tau_1^{(1)} > \tau_2^{(1)}, \quad \text{or} \quad \tau_1^{(1)} = \tau_2^{(1)} \text{and } \tau_1^{(2)} \geq \tau_2^{(2)} .$$

We then extend it to a total order on the set of admissible pairs, also denoted by $\succeq$. This induces a total order on the set $\{p_{\underline{\tau}}\}$: $p_{\underline{\tau}_1} \prec p_{\underline{\tau}_2} \Longleftrightarrow \underline{\tau}_1 \succ \underline{\tau}_2$ (we have taken the reverse order for a specific purpose). A monomial $\mathbf{m}$ of degree r in the polynomial ring $A = k[\{p_{\underline{\tau}}\}]$ will be written in the form $\mathbf{m} = p_{\underline{\tau}_1} p_{\underline{\tau}_2} \dots p_{\underline{\tau}_r}$, with $\underline{\tau}_1 \succeq \underline{\tau}_2 \succeq \dots \succeq \underline{\tau}_r$. As in §1.1, we consider the reverse lexicographic order on the set of monomials $\mathbf{m} \in k[\{p_{\underline{\tau}}\}]$, denoted by $\preceq_{rlex}$.

§3. Gröbner Basis for G/P

Definition 3.1 A monomial $\mathbf{m} = p_{\underline{\tau}_1} p_{\underline{\tau}_2} \dots p_{\underline{\tau}_r}$ of degree $r \in \mathbb{Z}^+$, where $\underline{\tau}_1 \succeq \underline{\tau}_2 \succeq \dots \succeq \underline{\tau}_r$ are admissible pairs, is called *standard* if $\tau_i^{(2)} \geq \tau_{i+1}^{(1)}$, $i \leq i \leq r-1$. If $\mathbf{m}$ is nonstandard, any pair $(i, i+1)$, $1 \leq i, i+1 \leq r$ such that $\tau_i^{(2)} \not\geq \tau_{i+1}^{(1)}$ is called a *violation of standardness in* $\mathbf{m}$.

Theorem 3.2 (cf. [LMS], [LS]$_2$) Standard monomials of degree r on G/P form a basis of $H^0(G/P, L^r)$.

As a consequence we obtain the following (cf. [iv], [v]):

Theorem 3.3 (cf. [LMS], [LS]$_2$) With the above notations, we have

(i) $R = \bigoplus H^0(G/P, L^r)$.

(ii) The restriction maps $\phi_r : \mathcal{S}^r(H^0(G/P,L)) \to H^0(G/P, L^r)$ are surjective, and induce a graded epimorphism $\phi : \mathcal{S}^\bullet(H^0(G/P,L)) \to R$.

Let $I = \ker \phi$; then I is a graded ideal and $I_r = \ker \phi_r$. We have $R = A/I$.

3.4 The set $\mathcal{F}$.

If $\mathbf{n}$ is a nonstandard monomial of degree $r \geq 2$, then, by Theorem 3.2, $\mathbf{n}$ can be written in a unique way as a linear combination of standard monomials of degree r, modulo the ideal I:

$$(*) \qquad \mathbf{n} = \sum_{i=1}^{t} a_i \mathbf{s}_i \pmod{I} , \qquad a_i \in k^*.$$

We refer to $(*)$ as a *straightening relation.* Denote

$$\mathbf{f_n} = \mathbf{n} - \sum_{i=1}^{t} a_i \mathbf{s}_i ,$$

$$\mathcal{F}_r = \{\mathbf{f_n} \mid \mathbf{n} \text{ is a nonstandard monomial of degree } r\}$$

and

$$\mathcal{F} = \bigcup_{r\geq 2} \mathcal{F}_r.$$

Clearly, $\mathcal{F}_r \subset I_r$ and $\mathcal{F} \subset I$.

$\mathcal{F}_2$ is going to play an important role, due to the following description of its elements (cf. [LMS]):

Theorem 3.5 Let

$$\mathbf{f}_{\underline{\tau}_1,\underline{\tau}_2} = p_{\underline{\tau}_1}p_{\underline{\tau}_2} - \sum_{i=1}^{t} a_i p_{\underline{\lambda}_i} p_{\underline{\nu}_i}, \qquad a_i \in k^*$$

be a typical element in $\mathcal{F}_2$, where $p_{\underline{\tau}_1}p_{\underline{\tau}_2}$ is a nonstandard monomial and $p_{\underline{\lambda}_i}p_{\underline{\nu}_i}$ are standard monomials of degree 2. Then:

(i) $\lambda_i^{(1)} > \tau_j^{(1)}$, for $j = 1,2$ and $1 \leq i \leq t$

(ii) $p_{\underline{\tau}_1}p_{\underline{\tau}_2}$ is the leading monomial of $\mathbf{f}_{\underline{\tau}_1,\underline{\tau}_2}$.

As a consequence, we obtain the following

Theorem 3.6 Let

$$\mathbf{f}_{\mathbf{n}} = \mathbf{n} - \sum_{i=1}^{t} a_i \mathbf{s}_i, \qquad a_i \in k^*$$

be a typical element in $\mathcal{F}$, where $\mathbf{n}$ is a nonstandard monomial and $\mathbf{s}_i$ are standard monomials, all of the same degree. Then lead($\mathbf{f}_{\mathbf{n}}$)=$\mathbf{n}$.

Proof. We proceed by induction with respect to the reverse lexicographic order. Let $\mathbf{n} = p_{\underline{\tau}_1}p_{\underline{\tau}_2}\cdots p_{\underline{\tau}_r}$, with $\underline{\tau}_1 \succeq \underline{\tau}_2 \succeq \ldots \succeq \underline{\tau}_r$, be a nonstandard monomial of degree r. Theorem 4 gives the starting point of induction. Let $(i, i+1)$ a violation of standardness in $\mathbf{n}$. Then $\mathbf{n}$ can be written in the form $\mathbf{n} = \mathbf{m}' p_{\underline{\tau}_i} p_{\underline{\tau}_{i+1}} \mathbf{m}''$, for some monomials $\mathbf{m}'$, $\mathbf{m}''$ and

$$\mathbf{m}'\mathbf{f}_{\underline{\tau}_1,\underline{\tau}_2}\mathbf{m}'' = \mathbf{n} - \sum_{i=1}^{t} a_i \mathbf{m}_i$$

where $\mathbf{m}_i = \mathbf{m}' p_{\underline{\lambda}_i} p_{\underline{\nu}_i} \mathbf{m}''$, with $\underline{\lambda}_i$, $\underline{\nu}_i$ as in Theorem 4. By Theorem 4, $\mathbf{m}_i \prec_{rlex} \mathbf{n}$ for any i. Using the induction hypothesis, each nonstandard monomial among the $\mathbf{m}_i$'s can be written as a linear combination of standard monomials which are strictly less than $\mathbf{m}_i$ in the reverse lexicographic order, hence less than $\mathbf{n}$ (since $\mathbf{m}_i \prec_{rlex} \mathbf{n}$ for any i). The result now follows from this.

Theorem 3.7 $\mathcal{F}$ is a Gröbner basis for I with respect to the monomial order $\preceq_{rlex}$.

Proof. We have to show that $\langle \text{lead}(\mathcal{F}) \rangle = \langle \text{lead}(I) \rangle$. We shall, in fact, prove that, $\text{lead}(\mathcal{F}) = \text{lead}(I)$. Since $\text{lead}(\mathcal{F}) \subset \text{lead}(I)$ and $\text{lead}(\mathcal{F})$ consists of all nonstandard monomials, it is enough to prove that the leading monomial of any element $f \in I$ is nonstandard. Assume this is not true, and let f be an element of I such that lead(f)

is a standard monomial. Let $\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_t$ be all the standard monomials, including $\mathrm{lead}(f) = \mathbf{s}_0$, and $\mathbf{n}_1, \ldots, \mathbf{n}_l$ all the nonstandard monomials appearing in f, so that f is written as

$$f = a_0\mathbf{s}_0 + \sum_{i=1}^{t} a_i\mathbf{s}_i + \sum_{j=1}^{l} b_j\mathbf{n}_j \ , \qquad a_0, a_i, b_j \in k^*.$$

Consider the polynomial $f' = f - \sum_{j=1}^{l} b_j\mathbf{f}_{\mathbf{n}_j}$ ($\mathbf{f}_{\mathbf{n}_j}$ being as in § 3.4). Then $\mathrm{lead}(f')$=$\mathbf{s}_0$, since $\mathrm{lead}(\mathbf{f}_{\mathbf{n}_j})$=$\mathbf{n}_j$, and $\mathbf{n}_j \prec_{rlex} \mathbf{s}_0$ for any j. Therefore f' is a nontrivial linear combination of standard monomials, since the coefficient of $\mathbf{s}_0$ in its expression is $a_0 \neq 0$. On the other hand $f' \equiv 0 \pmod I$, contradicting the linear independence of standard monomials assured by Theorem 2. Hence the assumption is wrong, and $\mathrm{lead}(f)$ is nonstandard.

In general, $\mathcal{F}$ is neither reduced, nor minimal.

Main Theorem I $\mathcal{F}_2$ is the reduced Gröbner basis for I with respect to the monomial order $\preceq_{rlex}$.

Proof. By Theorem 3.7, in order to prove that $\mathcal{F}_2$ is a Gröbner basis it is enough to show that $\langle \mathrm{lead}(\mathcal{F}_2)\rangle = \langle \mathrm{lead}(\mathcal{F})\rangle$. Since $\langle \mathrm{lead}(\mathcal{F}_2)\rangle \subset \langle \mathrm{lead}(\mathcal{F})\rangle$, it suffices to show that for any nonstandard monomial $\mathbf{n}$, $\mathrm{lead}(\mathbf{f}_\mathbf{n}) \in \langle \mathrm{lead}(\mathcal{F}_2)\rangle$, i.e. $\mathbf{n} \in \langle \mathrm{lead}(\mathcal{F}_2)\rangle$.

Let $\mathbf{n} = p_{\underline{\tau}_1} p_{\underline{\tau}_2} \ldots p_{\underline{\tau}_r}$ be nonstandard of degree r and $(i, i+1)$ a violation of standardness in $\mathbf{n}$, so that $p_{\underline{\tau}_i} p_{\underline{\tau}_{i+1}}$ is nonstandard. Since $\mathbf{n} \in \langle p_{\underline{\tau}_i} p_{\underline{\tau}_{i+1}}\rangle$ and $p_{\underline{\tau}_i} p_{\underline{\tau}_{i+1}} = \mathrm{lead}(\mathbf{f}_{\underline{\tau}_i, \underline{\tau}_{i+1}})$, we coclude that $\mathbf{n} \in \langle \mathrm{lead}(\mathcal{F}_2)\rangle$, which shows that $\mathcal{F}_2$ is a Gröbner basis. The fact that $\mathcal{F}_2$ is reduced can be easily seen from the form of its elements.

Remark 3.8 This theorem also implies $\mathcal{F}_2$ generates I as an ideal.

§4. Gröbner Bases for Schubert Varietes

Fix $w \in W^P$ and let $X(w)$ be the Schubert variety in G/P corresponding to w. Note that $G/P = X(w_P)$, where w_P is the minimal representative of w_0W_P, w_0 being the longest element in W. Denote by $R(w)$ the homogeneous coordinate ring of $X(w)$ for the embedding $X(w) \hookrightarrow \mathrm{Proj}(H^0(G/P, L))$. By Theorem 2.2, for any admissible pair $\underline{\tau}$ the coordinate function $p_{\underline{\tau}}$ is not identically zero on $X(w)$ if and only if $w \geq \tau^{(1)}$.

Definition 4.1 A *standard monomial on* $X(w)$ of degree r is a standard monomial of degree r on G/P of the form $\mathbf{m} = p_{\underline{\tau}_1} p_{\underline{\tau}_2} \ldots p_{\underline{\tau}_r}$, with $w \geq \tau_1^{(1)}$.

Thus the restriction to $X(w)$ of a standard monomial on G/P is either zero or standard on $X(w)$. Results similar to those on G/P hold on $X(w)$. We recall (cf.[LMS], [LS]$_2$):

Theorem 4.2 Standard monomials on $X(w)$ of degree r form a basis of $H^0(X(w), L^r)$.

Theorem 4.3 With the obove notations, we have

(i) The restriction map $H^0(G/P, L^r) \to H^0(X(w), L^r)$ is surjective.

(ii) $R(w) = \bigoplus H^0(X(w), L^r)$.

(iii) We have an epimorphism $R \to R(w)$ whose kernel is generated by $\{p_{\underline{\tau}} | w \not\geq \tau^{(1)}\}$.

Denote $A(w) = k[\{p_{\underline{\tau}} | w \geq \tau^{(1)}\}]$ and consider the canonical epimorphism $A \to A(w)$. Denote its kernel by $J(w)$; then $J(w)$ is generated by $\{p_{\underline{\tau}} | w \not\geq \tau^{(1)}\}$. By Theorem 4.3, we obtain an epimorphism $A(w) \to R(w)$, whose kernel is $I + J(w) \pmod{J(w)}$. We shall denote this kernel by $I(w)$; thus $R(w) = A(w)/I(w)$.

If $\mathbf{n}$ is a nonstandard monomial on $X(w)$ of degree r, then we denote the class of $\mathbf{f_n}$ in $A(w)$ by $\mathbf{f}^w_{\mathbf{n}}$. Let

$$\mathcal{F}^w_r = \{\mathbf{f}^w_{\mathbf{n}} | \ \mathbf{n} \text{ is a nostandard monomial on } X(w) \text{ of degree } r\ \}$$

and

$$\mathcal{F}^w = \bigcup_{r \geq 2} \mathcal{F}^w_r.$$

Clearly, $\mathcal{F}^w_r \subset I(w)_r$ and $\mathcal{F}^w \subset I(w)$.

The elements of $\mathcal{F}^w_2$ can be described as follows (cf. [LMS]):

Theorem 4.4 Let

$$\mathbf{f}^w_{\underline{\tau}_1, \underline{\tau}_2} = p_{\underline{\tau}_1} p_{\underline{\tau}_2} - \sum_{i=1}^{t} a_i p_{\underline{\lambda}_i} p_{\underline{\nu}_i}, \qquad a_i \in k^*$$

be a typical element in $\mathcal{F}^w_2$, where $p_{\underline{\tau}_1} p_{\underline{\tau}_2}$ is a nonstandard monomial and $p_{\underline{\lambda}_i} p_{\underline{\nu}_i}$ are standard monomials on $X(w)$ of degree 2. Then:

(i) $\lambda^{(1)}_i > \tau^{(1)}_j$ for $j = 1, 2$ and $1 \leq i \leq t$

(ii) $p_{\underline{\tau}_1} p_{\underline{\tau}_2}$ is the leading monomial of $\mathbf{f}^w_{\underline{\tau}_1, \underline{\tau}_2}$.

The following results follow from the discussion on G/P by simply restricting monomials to $X(w)$.

Theorem 4.5 Let

$$\mathbf{f}^w_{\mathbf{n}} = \mathbf{n} - \sum_{i=1}^{t} a_i \mathbf{s}_i, \qquad a_i \in k^*$$

be a typical element in $\mathcal{F}^w_r$, where $\mathbf{n}$ is a nonstandard monomial and $\mathbf{s}_i$ are standard monomials on $X(w)$, all of the same degree. Then lead($\mathbf{f}^w_{\mathbf{n}}$)=$\mathbf{n}$.

Theorem 4.6 $\mathcal{F}^w$ is a Gröbner basis for $I(w)$ with respect to the monomial order $\preceq_{rlex}$.

As in the previous section, we have:

Main Theorem II. $\mathcal{F}^w_2$ is the reduced Gröbner basis for $I(w)$ with respect to the monomial order $\preceq_{rlex}$.

§5. Grobner Bases For Varieties Arising in Classical Invariant Theory

In the discussion below, for the action of a reductive group G on an affine variety $X =$ Spec R, $X//G$ shall denote the categorical quotient Spec R^G.

I. $G = GL_n$

Let V be an n-dimensional k-vector space, and $G = GL(V)$. Let $X = V \oplus \cdots \oplus V \oplus V^* \oplus \cdots \oplus V^*$, m copies each, m being an integer $> n$ (here, V^* denotes the linear dual of V). For $x \in X$, say $x = (x_1, \cdots, x_m, f_1, \cdots, f_m)$, let $\varphi(x) = || < x_i, f_j > || \in M_m$ (= the space of $m \times m$ matrices), $<,>$ being the canonical bilinear form on $V \times V^*$. Then the morphism $\varphi : X \to M_m$ is G-invariant, φ maps X onto the determinantal variety D_n in M_m (defined by the vanishing of all $n \times n$ minors). Further, φ identifies $X//G$ with D_n. Now M_m can be identified with the opposite big cell $B^- e_{\text{id}}$ in $\text{Grass}_{m,2m}$, and D_n with the opposite big cell in a suitable Schubert variety Y in $\text{Grass}_{m,2m}$, i.e. $D_n \cong Y \cap B^- e_{\text{id}}$, where B^- denotes the Borel subgroup opposite to B (see $[\text{LS}]_1$ for details).

II. $G = O_{2n}$

Let V be a $2n$-dimensional k vector space together with a non-degenerate, and symmetric bilinear form (,), and $G = O(V)$. Let $X = V \oplus \cdots \oplus V$, m copies, m being an integer $> n$ For $x \in X$, say $x = (x_1, \cdots, x_m)$, let $\varphi(x) = || < x_i, x_j > || \in$ Sym M_m (= the space of symmetric $m \times m$ matrices). Then the morphism $\varphi : X \to$ Sym M_m is G-invariant, φ maps X onto the determinantal variety D_{2n} in Sym M_m . Further, φ identifies $X//G$ with D_{2n}. Now Sym M_m can be identified with the opposite big cell $B^- e_{\text{id}}$ in Sp_{2m}/P_m, P_m being the maximal parabolic subgroup corresponding to the right end root, and D_{2n} with the opposite big cell in a suitable Schubert variety in Sp_{2m}/P_m (see $[\text{LS}]_1$ for details).

III. $G = Sp_{2n}$

Let V be a $2n$-dimensional k vector space together with a non-degenerate, skew symmetric bilinear form (,), and $G = Sp(V)$. Let $X = V \oplus \cdots \oplus V$, m copies, m being an integer $> n$. For $x \in X$, say $x = (x_1, \cdots, x_m)$, let $\varphi(x) = || < x_i, x_j > || \in$ Sk M_m (= the space of skew symmetric $m \times m$ matrices). Then the morphism $\varphi : X \to$ Sym M_m is G-invariant, φ maps X onto the determinantal variety D_{2n} in Sk M_m . Further, φ identifies $X//G$ with D_{2n}. Now Sk M_m can be identified with the opposite big cell $B^- e_{\text{id}}$ in SO_{2m}/P_m, P_m being the maximal parabolic subgroup corresponding to one of the right end roots, and D_{2n} with the opposite big cell in a suitable Schubert variety in SO_{2m}/P_m (see $[\text{LS}]_1$ for details).

In all of the above three cases, R^G gets identified with the homogeneous localization at p_{id} of the homogeneous coordinate ring of the corresponding Schubert variety. Now p_{id} is the largest among $\{p_{\underline{\tau}}\}$, and hence specializing, p_{id} to 1 in any standard (resp. non-standard) monomial $\mathbf{m}$, the resulting monomial $\mathbf{m}'$ remains standard (resp. non-standard). Hence using the results of §4, we obtain reduced Grobner bases for these varieties.

§6. Concluding Remarks

1. Reduced Gröbner bases for Schubert varietes in G/P, for P of non-classical type can be constructed using [L]. More generally, reduced Gröbner bases for Schubert varietes (and also for unions of Schubert varietes) in G/Q, Q being *any* parabolic subgroup, can be constructed using [L]. The details will appear elsewhere.

2. In [MS], it is shown that a variety of complexes can be identified with $B^- e_{\mathrm{id}} \cap X$, where X is a union of Schubert varietes in SL_n/Q, for a suitable n, and a suitable parabolic subgroup. Hence, we obtain reduced Gröbner bases for a variety of complexes.

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