# A GEOMETRIC APPROACH TO STANDARD MONOMIAL THEORY

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Abstract. We obtain a geometric construction of a "standard monomial basis" for the homogeneous coordinate ring associated with any ample line bundle on any flag variety. This basis is compatible with Schubert varieties, opposite Schubert varieties, and unions of intersections of these varieties. Our approach relies on vanishing theorems and a degeneration of the diagonal ; it also yields a standard monomial basis for the multi–homogeneous coordinate rings of flag varieties of classical type.

## Introduction

Consider the Grassmannian $X$ of linear subspaces of dimension $r$ in $k^n$, where $k$ is a field. We regard $X$ as a closed subvariety of projective space $\mathbb{P}(\wedge^r k^n)$ via the Plücker embedding; let $L$ be the corresponding very ample line bundle on $X$. Then the ring $\bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m})$ admits a nice basis, defined as follows.

Let $\{v_1, \ldots, v_n\}$ be the usual basis of $k^n$; then the $v_{i_1} \wedge \cdots \wedge v_{i_r}$, $1 \le i_1 < \cdots < i_r \le n$, form a basis of $\wedge^r k^n$. We put $I = (i_1, \ldots, i_r)$, $v_I = v_{i_1} \wedge \cdots \wedge v_{i_r}$, and we denote by $\{p_I\}$ the dual basis of the basis $\{v_I\}$; the $p_I$ (regarded in $H^0(X, L)$) are the Plücker coordinates. Define a partial order on the set $\mathcal{I}$ of indices $I$ by letting $I = (i_1, \ldots, i_r) \le (j_1, \ldots, j_r) = J$ if and only if $i_1 \le j_1, \ldots, i_r \le j_r$. Then

(i) *The monomials* $p_{I_1} p_{I_2} \cdots p_{I_m}$ *where* $I_1, \ldots, I_m \in \mathcal{I}$ *satisfy* $I_1 \le I_2 \le \cdots \le I_m$, *form a basis of* $H^0(X, L^{\otimes m})$.

(ii) *For any* $I, J \in \mathcal{I}$, *we have* $p_I\, p_J - \sum_{I',J',\, I' \le I, J \le J'} a_{I'J'}\, p_{I'}\, p_{J'} = 0$, *where* $a_{I'J'} \in k$.

The monomials in (i) are called the *standard monomials of degree* $m$, and the relations in (ii) are the *quadratic straightening relations*; they allow to express any non–standard monomial in the $p_I$ as a linear combination of standard monomials.

Further, this *standard monomial basis* of the homogeneous coordinate ring of $X$ is compatible with its Schubert subvarieties, in the following sense. For any $I \in \mathcal{I}$, let $X_I = \{V \in X \mid \dim(V \cap \operatorname{span}(v_1, \ldots, v_s)) \ge \#(j, i_j \le s), 1 \le s \le r\}$ be the corresponding Schubert variety; then the restriction $p_J|_{X_I}$ is nonzero if and only if $J \le I$. The monomial $p_{I_1} \cdots p_{I_m}$ will be called *standard on* $X_I$ if $I_1 \le \cdots \le I_m \le I$; equivalently, this monomial is standard and does not vanish identically on $X_I$. Now

(iii) *The standard monomials of degree m on $X_I$ restrict to a basis of $H^0(X_I, L^{\otimes m})$. The standard monomials of degree m that are not standard on $X_I$, form a basis of the kernel of the restriction map $H^0(X, L^{\otimes m}) \to H^0(X_I, L^{\otimes m})$.*

These classical results go back to Hodge, see [5]. They have important geometric consequences, e.g., $X$ is projectively normal in the Plücker embedding; its homogeneous ideal is generated by the quadratic straightening relations; the homogeneous ideal of any Schubert variety $X_I$ is generated by these relations together with the $p_J$ where $J \not\leq I$.

The purpose of *Standard Monomial Theory* (SMT) is to generalize Hodge's results to any flag variety $X = G/P$ (where $G$ is a semisimple algebraic group over an algebraically closed field $k$, and $P$ a parabolic subgroup) and to any effective line bundle $L$ on $X$. SMT was developed by Lakshmibai, Musili, and Seshadri in a series of papers, culminating in [9] where it is established for all classical groups $G$. There the approach goes by ascending induction on the Schubert varieties, using their partial resolutions as projective line bundles over smaller Schubert varieties.

Further results concerning certain exceptional or Kac–Moody groups led to conjectural formulations of a general SMT, see [10]. These conjectures were then proved by Littelmann, who introduced new combinatorial and algebraic tools: the path model of representations of any Kac–Moody group, and Lusztig's Frobenius map for quantum groups at roots of unity (see [11, 12]).

In the present paper, we obtain a geometric construction of a SMT basis for $H^0(X, L)$, where $X = G/P$ is any flag variety and $L$ is any ample line bundle on $X$. This basis is compatible with Schubert varieties (that is, with orbit closures in $X$ of a Borel subgroup $B$ of $G$) and also with opposite Schubert varieties (the orbit closures of an opposite Borel subgroup $B^-$); in fact, it is compatible with any intersection of a Schubert variety with an opposite Schubert variety. We call such intersections *Richardson varieties*, since they were first considered by Richardson in [17]. Our approach adapts to the case where $L$ is an effective line bundle on a flag variety *of classical type* in the sense of [9]. This sharpens the results of [9] concerning the classical groups.

Our work may be regarded as one step towards a purely geometric proof of Littelmann's results concerning SMT. He constructed a basis of $T$–eigenvectors for $H^0(X, L)$ (where $T$ is the maximal torus common to $B$ and $B^-$) indexed by certain piecewise linear paths in the positive Weyl chamber, called *LS paths*. This basis turns out to be compatible with Richardson varieties; notice that these are $T$–invariant. In fact, the endpoints of the path indexing a basis vector parametrize the smallest Richardson variety where this vector does not vanish identically (see [8]). If $L$ is associated with a weight of classical type, then the LS paths are just line segments:

they are uniquely determined by their endpoints. This explains a posteriori why our geometric approach completes the program of SMT in that case.

In fact, our approach of SMT for an ample line bundle $L$ on a flag variety $X$ uses little of the rich geometry and combinatorics attached to $X$. Specifically, we only rely on vanishing theorems for unions of Richardson varieties (these being direct consequences of the existence of a Frobenius splitting of $X$, compatible with Schubert varieties and opposite Schubert varieties), together with the following property.

(iv) *The diagonal in $X \times X$ admits a flat $T$–invariant degeneration to the union of all products $X_w \times X^w$, where the $X_w$ are the Schubert varieties and the $X^w$ are the corresponding opposite Schubert varieties.*

The latter result follows from [2] (we provide a direct proof in Section 3). It plays an essential rôle in establishing generalizations of (i) and (iii); conversely, it turns out that the existence of a SMT basis implies (iv), see the Remark after Proposition 7.

It is worth noticing that (iv) is a stronger form of the fact that the classes of Schubert varieties form a free basis of the homology group (or Chow group) of $X$, the dual basis for the intersection pairing consisting of the classes of opposite Schubert varieties. This fact (in a different formulation) has been used by Knutson to establish an asymptotic version of the Littelmann character formula, see [7].

This paper is organized as follows. In the preliminary Section 1, we introduce notation and study the geometry of Richardson varieties. Vanishing theorems for cohomology groups of line bundles on Richardson varieties are established in Section 2, by slight generalizations of the methods of Frobenius splitting. In Section 3, we construct filtrations of the $T$–module $H^0(X, L)$ that are compatible with restrictions to Richardson varieties. Our SMT basis of $H^0(X, L)$ is defined in Section 4; it is shown to be compatible with all unions of Richardson varieties. In Section 5, we generalize statements (i) and (iii) above to any ample line bundle $L$ on a flag variety $G/P$; then (ii) follows from (i) together with compatibility properties of our basis. The case where the homogeneous line bundle $L$ is associated with a weight of classical type (e.g., a fundamental weight of a classical group) is considered in detail in Section 6. There we give a geometric characterization of the *admissible pairs* of [9] (these parametrize the weights of the $T$–module $H^0(X, L)$). The final Section 7 develops SMT for those effective line bundles that correspond to sums of weights of classical type.

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## 1. Richardson varieties

The ground field $k$ is algebraically closed, of arbitrary characteristic. Let $G$ be a simply–connected semisimple algebraic group. Choose opposite Borel subgroups $B$ and $B^-$ of $G$, with common torus $T$; let $\mathcal{X}(T)$ be the group of characters of $T$, also called weights. In the root system $R$ of $(G,T)$, we have the subset $R^+$ of positive roots (that is, of roots of $(B,T)$), and the subset $S$ of simple roots. For each $\alpha \in R$, let $\check{\alpha}$ be the corresponding coroot and let $U_\alpha$ be the corresponding additive one–parameter subgroup of $G$, normalized by $T$.

We also have the Weyl group $W$ of $(G,T)$; for each $\alpha \in R$, we denote by $s_\alpha \in W$ the corresponding reflection. Then the group $W$ is generated by the simple reflections $s_\alpha$, $\alpha \in S$; this defines the length function $\ell$ and the Bruhat order $\leq$ on $W$. Let $w_o$ be the longest element of $W$, then $B^- = w_o B w_o$.

Let $P$ be a parabolic subgroup of $G$ containing $B$ and let $W_P$ be the Weyl group of $(P,T)$, a parabolic subgroup of $W$; let $w_{o,P}$ be the longest element of $W_P$. Each right $W_P$–coset in $W$ contains a unique element of minimal length; this defines the subset $W^P$ of minimal representatives of the quotient $W/W_P$. This subset is invariant under the map $w \longmapsto w_o w w_{o,P}$; the induced bijection of $W^P$ reverses the Bruhat order.

Each character $\lambda$ of $P$ defines a $G$–linearized line bundle on the homogeneous space $G/P$; we denote that line bundle by $L_\lambda$. The assignement $\lambda \longmapsto L_\lambda$ yields an isomorphism from the character group $\mathcal{X}(P)$ to the Picard group of $G/P$. Further, the line bundle $L_\lambda$ is generated by its global sections if and only if $\lambda$ (regarded as a character of $T$) is dominant; in that case, $H^0(G/P, L_\lambda)$ is a $G$–module with lowest weight $-\lambda$.

Let $W_\lambda$ be the isotropy group of $\lambda$ in $W$, and let $P_\lambda$ be the parabolic subgroup of $G$ generated by $B$ and $W_\lambda$; then $W_\lambda \supseteq W_P$, $W^\lambda \subseteq W^P$, and $P_\lambda \supseteq P$. We shall identify $W^\lambda$ with the $W$–orbit of the weight $\lambda$, and denote by $w(\lambda)$ the image of $w \in W$ in $W/W_\lambda \simeq W^\lambda$.

The *extremal weight vectors* $p_{w(\lambda)} \in H^0(G/P, L_\lambda)$ are the $T$–eigenvectors of weight $-w(\lambda)$ for some $w \in W^\lambda$. These vectors are uniquely defined up to scalars.

We say that $\lambda$ is *$P$–regular* if $P_\lambda = P$. The ample line bundles on $G/P$ are the $L_\lambda$ where $\lambda$ is dominant and $P$–regular; under these assumptions, $L_\lambda$ is in fact very ample. We may then identify each $w \in W^P$ to $w(\lambda)$, and we put $p_w = p_{w(\lambda)}$.

The $T$–fixed points in $G/P$ are the $e_w = wP/P$ ($w \in W/W_P$); we index them by $W^P$. The $B$–orbit $C_w = Be_w$ is a *Bruhat cell*, an affine space of dimension $\ell(w)$; it closure in $G/P$ is the *Schubert variety* $X_w$. The complement $X_w - C_w$ is the *boundary* $\partial X_w$. We have

$$\partial X_w = \bigcup_{v \in W^P,\, v < w} X_v,$$

and the irreducible components of $\partial X_w$ are the *Schubert divisors* $X_v$ where $v \in W^P$, $v < w$ and $\ell(v) = \ell(w) - 1$. Then there exists $\beta \in R^+$ such that $v = ws_\beta$.

Let $\lambda$ be a character of $P$ and let $f_w$ be the restriction to $X_w$ of the natural map $G/P \longrightarrow G/P_\lambda$; then $f_w(X_w) = X_{w(\lambda)}$. The set

$$\partial_\lambda X_w := f_w^{-1}(\partial X_{w(\lambda)})$$

is called the $\lambda$–*boundary* of $X_w$; it is the union of the Schubert divisors $X_{ws_\beta}$ where $\langle \lambda, \check{\beta} \rangle > 0$. If $\lambda$ is dominant, then we have by Chevalley's formula:

$$\operatorname{div}(p_{w(\lambda)}|_{X_w}) = \sum \langle \lambda, \check{\beta} \rangle X_{ws_\beta}$$

(sum over all $\beta \in R^+$ such that $X_{s_\beta w}$ is a divisor in $X_w$). In particular, the zero set of $p_{w(\lambda)}$ in $X_w$ is $\partial_\lambda X_w$. If in addition $\lambda$ is $P$–regular, then $\partial_\lambda X_w = \partial X_w$.

We shall also need the *opposite Bruhat cell* $C^w = B^- e_w$ of codimension $\ell(w)$ in $G/P$, the *opposite Schubert variety* $X^w$ (the closure of $C^w$) and its boundary $\partial X^w$. Then $X^w = w_o X_{w_o w w_{o,P}}$ and

$$\partial X^w = \bigcup_{v \in W^P, v > w} X^v.$$

Recall that all Schubert varieties are normal and Cohen–Macaulay (thus, the same holds for all opposite Schubert varieties). Further, all scheme–theoretic intersections of unions of Schubert varieties and opposite Schubert varieties are reduced (see [14, 15, 16]).

**Definition 1.** *Let $v, w$ in $W^P$. We call the intersection*

$$X_w^v := X_w \cap X^v$$

*a Richardson variety in $G/P$. We define its boundaries by*

$$(\partial X_w)^v := \partial X_w \cap X^v \text{ and } (\partial X^v)_w := X_w \cap \partial X^v.$$

Notice that $X_w^v$ and its boundaries are closed reduced, $T$–stable subschemes of $G/P$. The $X_w^v$ were considered by Richardson, who showed e.g. that they are irreducible (see [17]; the intersections $C_w \cap C^v$ were analyzed by Deodhar, see [4]). We shall give another proof of this result, and obtain a little more.

**Lemma 1.** (1) *$X_w^v$ is non–empty if and only if $v \le w$; then $X_w^v$ is irreducible of dimension $\ell(w) - \ell(v)$, and $(\partial X_w)^v$, $(\partial X^v)_w$ have pure codimension 1 in $X_w^v$. Further, $X_w^v$ is normal and Cohen–Macaulay.*

(2) *The $T$–fixed points in $X_w^v$ are the $e_x$ where $x \in W^P$ and $v \le x \le w$.*

(3) *For $x, y$ in $W^P$, we have $X_y^x \subseteq X_w^v \iff v \le x \le y \le w$.*

*Proof.* (2) is evident; it implies (3) and the first assertion of (1). To prove the remaining assertions, we use a variant of the argument of [1] Lemma 2. Consider the fiber product $G \times^B X_w$ with projection map

$$p : G \times^B X_w \longrightarrow G/B,$$

a $G$–equivariant locally trivial fibration with fiber $X_w$. We also have the "multiplication" map

$$m : G \times^B X_w \longrightarrow G/P,\ (g,x) \longmapsto gx.$$

This is a $G$–equivariant map to $G/P$; thus, it is also a locally trivial fibration. Its fiber $m^{-1}(e_1)$ is isomorphic to $\overline{Pw^{-1}B}/B$ (a Schubert variety in $G/B$).

Next let $i : X^v \longrightarrow G/P$ be the inclusion and consider the cartesian product

$$Z = X^v \times_{G/P} (G \times^B X_w)$$

with projections $\iota$ to $G \times^B X_w$, $\mu$ to $X^v$ and $\pi$ to $G/B$, as displayed in the following commutative diagram:

$$\begin{array}{ccccc} G/B & \xleftarrow{\pi} & Z & \xrightarrow{\mu} & X^v \\ {\scriptstyle id}\downarrow & & {\scriptstyle \iota}\downarrow & & {\scriptstyle i}\downarrow \\ G/B & \xleftarrow{p} & G \times^B X_w & \xrightarrow{m} & G/P \end{array}$$

By definition, the square on the right is cartesian, so that $\mu$ is also a locally trivial fibration with fiber $\overline{Pw^{-1}B}/B$ and base $X^v$. Since Schubert varieties are irreducible, normal and Cohen–Macaulay, it follows that the same holds for $Z$. Further, we have

$$\dim(Z) = \dim(G \times^B X_w) + \dim(X^v) - \dim(G/P) = \dim(G/B) + \ell(w) - \ell(v).$$

Notice that the fiber of $\pi : Z \longrightarrow G/B$ at each $gB/B$ identifies to the intersection $X^v \cap gX_w$; in particular, $\pi^{-1}(B/B) = X^v_w$. Notice also that $\iota : Z \longrightarrow G \times^B X_w$ is a closed immersion with $B^-$–stable image (since this holds for $i : X^v \longrightarrow G/P$). Thus, $B^-$ acts on $Z$ so that $\pi$ is equivariant. Since $B^-B/B$ is an open neighborhood of $B/B$ in $G/B$, isomorphic to $U^-$, its pullback under $\pi$ is an open subset of $Z$, isomorphic to $U^- \times X^v_w$. Therefore, $X^v_w$ is irreducible, normal and Cohen–Macaulay of dimension $\ell(w) - \ell(v)$. □

We also record the following easy result, to be used in Section 7.

**Lemma 2.** *Let $v \le w$ in $W^P$, let $\lambda$ be a dominant character of $P$ and let $x(\lambda) \in W^\lambda$. Then the restriction of $p_{x(\lambda)}$ to $X^v_w$ is non–zero if and only if $x(\lambda)$ admits a lift $x \in W^P$ such that $v \le x \le w$. Further, the ring*

$$\bigoplus_{n=0}^{\infty} H^0(X^v_w, L_{n\lambda})$$

*is integral over its subring generated by the $p_{x(\lambda)}|_{X^v_w}$ where $x \in W^P$ and $v \le x \le w$.*

*Proof.* Consider the natural map $G/P \to G/P_\lambda$ and its restriction $f : X_w^v \longrightarrow f(X_w^v)$. The open subset $(p_{x(\lambda)} \neq 0)$ of $G/P_\lambda$ is affine, $T$–stable and contains $e_{x(\lambda)}$ as its unique closed $T$–orbit. Thus, $p_{x(\lambda)}|_{X_w^v} \neq 0$ if and only if $e_{x(\lambda)} \in f(X_w^v)$. By Borel's fixed point theorem, this amounts to the existence of a $T$–fixed point $e_x \in X_w^v$ such that $f(e_x) = e_{x(\lambda)}$. Now Lemma 1 (2) completes the proof of the first assertion.

By the preceding arguments, the sections $p_{x(\lambda)}|_{X_w^v}$, $x \in W^P$, $v \leq x \leq w$ do not vanish simultaneously at a $T$–fixed point of $X_w^v$. Since these sections are eigenvectors of $T$, it follows that they have no common zeroes. This implies the second assertion. □

**Remark.** The image of a Richardson variety $X_w^v$ under a morphism $G/P \longrightarrow G/P_\lambda$ need not be another Richardson variety. Consider for example $G = SL(3)$ with simple reflections $s_1$, $s_2$. Let $P = B$, $w = s_2 s_1$, $v = s_2$ and $\lambda = \omega_1$ (the fundamental weight fixed by $s_2$). Then $X_w^v$ is one–dimensional and mapped isomorphically to its image $f(X_w^v)$ in $G/P_\lambda$. Since the $T$–fixed points in $f(X_w^v)$ are $e_{\omega_1}$ and $e_{s_2 s_1(\omega_1)}$, it follows that $f(X_w^v)$ is not a Richardson variety.

## 2. Cohomology vanishing for Richardson varieties

In this section, we assume that the characteristic of $k$ is $p > 0$. Let $X$ be a scheme of finite type over $k$. Let $F : X \longrightarrow X$ be the absolute Frobenius morphism, that is, $F$ is the identity map on the topological space of $X$, and $F^\# : \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$ is the $p$–th power map. Then $X$ is called *Frobenius split* if the map $F^\#$ is split. We shall need a slight generalization of this notion, involving the composition $F^r = F \circ \cdots \circ F$ ($r$ times), where $r$ is any positive integer.

**Definition 2.** *We say that $X$ is split if there exists a positive integer $r$ such that the map*

$$(F^r)^\# : \mathcal{O}_X \longrightarrow F_*^r \mathcal{O}_X$$

*splits, that is, there exists an $\mathcal{O}_X$–linear map*

$$\varphi : F_*^r \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

*such that $\varphi \circ (F^r)^\#$ is the identity; then $\varphi$ is called a splitting.*

We shall also need a slight generalization of the notion of Frobenius splitting relative to an effective Cartier divisor (see [16]).

**Definition 3.** *Let $X$ be a normal variety and $D$ an effective Weil divisor on $X$, with canonical section $s$. We say that $X$ is $D$–split if there exist a positive integer $r$ and an $\mathcal{O}_X$–linear map*

$$\psi : F_*^r \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$$

*such that the map*

$$\varphi : F_*^r \mathcal{O}_X \longrightarrow \mathcal{O}_X, \;\; f \longmapsto \psi(fs)$$

*is a splitting. Then $\psi$ is called a $D$–splitting.*

*We say that a closed subscheme $Y$ of $X$, with ideal sheaf $\mathcal{I}_Y$, is compatibly $D$–split if (a) no irreducible component of $Y$ is contained in the support of $D$, and (b) $\varphi(F^r_*\mathcal{I}_Y) = \mathcal{I}_Y$.*

**Remarks.** (i) Let $U$ be an open subset of $X$ such that $X - U$ has codimension at least 2 in $X$. Then $X$ is $D$–split if and only if $U$ is $D \cap U$–split (to see this, let $i : U \longrightarrow X$ be the inclusion, then $i_*\mathcal{O}_U = \mathcal{O}_X$ and $i_*\mathcal{O}_U(D \cap U) = \mathcal{O}_X(D)$ by normality of $X$.)

Let $Y$ be a closed subscheme of $X$ such that $Y \cap U$ is dense in $Y$. Then $Y$ is compatibly $D$–split if and only if $Y \cap U$ is compatibly $D \cap U$–split (this is checked by the arguments of [16] 1.4–1.7).

(ii) If $X$ is split compatibly with an effective Weil divisor $D$, then $X$ is $(p^r - 1)D$–split (to see this, one may assume that $X$ is nonsingular, by (i). Let $\varphi$ be a compatible splitting, then $\varphi(F^r_*\mathcal{O}_X(-D)) = \mathcal{O}_X(-D)$. Define $\psi : F^r_*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$ by $\psi(f\sigma^{p^r-1}) = \sigma\varphi(f\sigma^{-1})$ for any local sections $f$ of $\mathcal{O}_X$ and $\sigma$ of $\mathcal{O}_X(D)$. Then one checks that $\psi$ is well–defined, $\mathcal{O}_X$–linear and satisfies $\psi(fs^{p^r-1}) = \varphi(f)$.

(iii) Let $D$ and $E$ be effective Weil divisors in $X$, such that $D - E$ is effective. If $X$ is $D$–split, then it is $E$–split as well; if in addition a closed subscheme $Y$ of $X$ is compatibly $D$–split, then it is compatibly $E$–split (this follows from (i) together with [16] Remark 1.3 (ii).)

**Lemma 3.** *Let $D$, $E$ be effective Weil divisors on a normal variety $X$, such that the support of $D$ contains the support of $E$. If $X$ is $D$–split, then $X$ is $E$–split as well. If moreover a closed subscheme $Y$ of $X$ is compatibly $D$–split, then $X$ is compatibly $E$–split.*

*Proof.* Let $U$ be the set of those points of $X$ at which $D$ is a Cartier divisor. Then $U$ is an open subset with complement of codimension at least 2 (since $U$ contains the nonsingular locus of $X$). Moreover, $Y \cap U$ is dense in $Y$ (since $U$ contains the complement of the support of $D$). Thus, by Remark (i), we may replace $X$ with $U$, and hence assume that $D$ is a Cartier divisor.

Now let $\psi : F^r_*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$ be a $D$–splitting. We regard $\psi$ as an additive map $\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$ such that: $\psi(s) = 1$, and $\psi(f^{p^r}\sigma) = f\psi(\sigma)$ for any local sections $f$ of $\mathcal{O}_X$ and $\sigma$ of $\mathcal{O}_X(D)$. For any positive integer $n$, we set

$$\mathbf{n} = p^{r(n-1)} + p^{r(n-2)} + \cdots + 1$$

(then $\mathbf{1} = 1$), and we define inductively a map

$$\psi^n : F^{rn\#}\mathcal{O}_X(\mathbf{n}D) \longrightarrow \mathcal{O}_X$$

by: $\psi^1 = \psi$, and

$$\psi^n(f\sigma^{\mathbf{n}}) = \psi(\psi^{n-1}(f\sigma^{\mathbf{n-1}})\sigma)$$

for any local sections $f$ of $\mathcal{O}_X$ and $\sigma$ of $\mathcal{O}_X(D)$. Then one may check that $\psi^n$ is well defined and is a $\mathbf{n}D$–splitting of $X$. If moreover a closed subscheme $Y$ is compatibly $D$–split, then $\psi(F_*^r(\mathcal{I}_Y s)) = \mathcal{I}_Y$. By induction, it follows that $\psi^n(F_*^{rn}(\mathcal{I}_Y s^{\mathbf{n}})) = \mathcal{I}_Y$, so that $Y$ is compatibly $\mathbf{n}D$–split.

Since the support of $D$ contains the support of $E$, there exists a positive integer $n$ such that $\mathbf{n}D - E$ is effective. Then $X$ is $\mathbf{n}D$–split, so that it is $E$-split by Remark (ii). □

**Lemma 4.** *Let $X$ be a normal projective variety endowed with an effective Weil divisor $D$ and with a globally generated line bundle $L$; let $Y$ be a closed subscheme of $X$. Assume that (a) $X$ is $D$–split compatibly with $Y$, and (b) the support of $D$ contains the support of an effective ample divisor. Then $H^i(X, L) = 0 = H^i(Y, L)$ for all $i \geq 1$, and the restriction map $H^0(X, L) \longrightarrow H^0(Y, L)$ is surjective.*

*Proof.* Choose an effective ample Cartier divisor $E$, with support contained in the support of $D$. Then $X$ is $E$–split compatibly with $Y$, by Lemma 3. Now the assertions follow from [16] 1.12, 1.13. □

We now apply this to Richardson varieties. By [16] 3.5, the variety $G/P$ is split compatibly with all Schubert varieties and with all opposite Schubert varieties; as a consequence, $G/P$ is split compatibly with all unions of Richardson varieties. By [16] 1.10, it follows that all scheme–theoretical intersections of unions of Richardson varieties are reduced; and using [16] 1.13, this also implies

**Lemma 5.** *Let $\lambda$ be a regular dominant character of $P$ and let $Z$ be a union of Richardson varieties in $G/P$. Then the restriction map $H^0(G/P, L_\lambda) \longrightarrow H^0(Z, L_\lambda)$ is surjective, and $H^i(Z, L_\lambda) = 0$ for all $i \geq 1$. As a consequence, $H^i(X, L_\lambda \otimes \mathcal{I}_Z) = 0$ for all $i \geq 1$.*

**Remark.** If we only assume that $\lambda$ is dominant, then Lemma 5 extends to all unions of Schubert varieties (by [16]), but not to all unions of Richardson varieties. As a trivial example, take $G/P = \mathbb{P}^1$, the projective line with $T$–fixed points 0 and $\infty$, and $\lambda = 0$. Then $Z := \{0, \infty\}$ is a union of Richardson varieties, and the restriction map $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow H^0(Z, \mathcal{O}_Z)$ is not surjective. As a less trivial example, take $G/P = \mathbb{P}^1 \times \mathbb{P}^1$, $Z = (\mathbb{P}^1 \times \{0, \infty\}) \cup (\{0, \infty\} \times \mathbb{P}^1)$, and $\lambda = 0$. Then $Z$ is again a union of Richardson varieties, and one checks that $H^1(Z, \mathcal{O}_Z) \neq 0$.

However, Lemma 5 does extend to all dominant characters and to unions of Richardson varieties *with a common index.*

**Proposition 1.** *Let $\lambda$ be a dominant character of $P$ and let $Z$ be a union of Richardson varieties $X_w^v$ in $G/P$, all having the same $w$. Then the restriction $H^0(G/P, L_\lambda) \to H^0(Z, L_\lambda)$ is surjective, and $H^i(Z, L_\lambda) = 0$ for all $i \geq 1$.*

*As a consequence, we have $H^i(X_w^v, L_\lambda(-Z)) = 0$ for all $i \geq 1$, where $v \leq w$ in $W^P$, and $Z$ is a union of irreducible components of $(\partial X^v)_w$.*

*Proof.* The Schubert variety $X_w$ is split compatibly with the effective Weil divisor $\partial X_w$ and with $Z$. By assumption, $\partial X_w$ contains no irreducible component of $Z$. Using Remarks (i) and (ii), it follows that $X_w$ is $(p-1)\partial X_w$–split compatibly with $Z$. Further, $\partial X_w$ is the support of an ample effective divisor, as follows from Chevalley's formula. Thus, Lemma 4 applies and yields surjectivity of $H^0(X_w, L_\lambda) \longrightarrow H^0(Z, L_\lambda)$ together with vanishing of $H^i(Z, L_\lambda)$ for $i \geq 1$. Now surjectivity of $H^0(G/P, L_\lambda) \longrightarrow H^0(X_w, L_\lambda)$ completes the proof of the first assertion.

In particular, we have $H^i(X_w^v, L_\lambda) = H^i(Z, L_\lambda) = 0$ for all $i \geq 1$, and the restriction map $H^0(X_w^v, L_\lambda) \longrightarrow H^0(Z, L_\lambda)$ is surjective; this implies the second assertion. □

We shall also need the following, more technical vanishing result.

**Proposition 2.** *Let $\lambda$ be a dominant character of $P$ and let $v, w$ in $W^P$ such that $v \leq w$. Then*

$$H^i(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v - Z)) = 0$$

*for any $i \geq 1$ and for any (possibly empty) union $Z$ of irreducible components of $(\partial X^v)_w$.*

*Proof.* We shall rely on the following result (see [13] Theorem 1). Let $\pi : X \longrightarrow Y$ be a proper morphism of schemes. Let $D$ (resp. $E$) be a closed subscheme of $X$ (resp. $Y$) and let $i$ be a positive integer such that:
(i) $\pi^{-1}(E)$ is contained in $D$ (as sets).
(ii) $R^i\pi_*(\mathcal{I}_D) = 0$ outside $E$.
(iii) $X$ is split compatibly with $D$.
Then $R^i\pi_*(\mathcal{I}_D) = 0$ everywhere.

To apply this result, consider the restriction

$$f : X_w^v \longrightarrow f(X_w^v)$$

of the natural map $G/P \longrightarrow G/P_\lambda$. Then $L_\lambda = f^* M_\lambda$ for a very ample line bundle $M_\lambda$ on $f(X_w^v)$. Let $Y$ be the corresponding affine cone over $f(X_w^v)$, with vertex 0 and projection map

$$q : Y - \{0\} \longrightarrow f(X_w^v).$$

And let $X$ be the total space of the line bundle $L_{-\lambda}$ (dual to $L_\lambda$), with projection map

$$p : X \longrightarrow X_w^v$$

and zero section $X_0$. Then the algebra

$$H^0(X, \mathcal{O}_X) = \bigoplus_{n=0}^{\infty} H^0(X_w^v, L_{n\,\lambda})$$

contains $H^0(Y, \mathcal{O}_Y)$ as the subalgebra generated by $H^0(f(X_w^v), M_\lambda)$. The algebra $H^0(X, \mathcal{O}_X)$ is finitely generated, and the corresponding morphism

$$X \longrightarrow \text{Spec } H^0(X, \mathcal{O}_X)$$

is proper, since the line bundle $L_\lambda$ is globally generated. Moreover, since $L_\lambda$ is the pullback under $f$ of the very ample line bundle $M_\lambda$, the algebra $H^0(X, \mathcal{O}_X)$ is a finite module over its subalgebra $H^0(Y, \mathcal{O}_Y)$. This defines a proper morphism $\pi : X \longrightarrow Y$, and we have $\pi^{-1}(0) = X_0$ (as sets). Moreover, the diagram

$$\begin{array}{ccc} X - X_0 & \xrightarrow{\pi} & Y - \{0\} \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow \\ X_w^v & \xrightarrow{f} & f(X_w^v) \end{array}$$

is cartesian, and the vertical maps are principal $\mathbb{G}_m$–bundles.

Now let $D = X_0 \cup p^{-1}((\partial_\lambda X_w)^v \cup Z)$; this is a closed subscheme of $X$ with ideal sheaf

$$\mathcal{I}_D = p^* L_\lambda(-(\partial_\lambda X_w)^v - Z).$$

Let $E$ be the affine cone over $f((\partial_\lambda X_w)^v)$; this is a closed subscheme of $Y$. We check that the conditions (i), (ii) and (iii) hold.

For (i), notice that

$$\pi^{-1}(E) = X_0 \cup p^{-1} f^{-1} f((\partial_\lambda X_w)^v) = X_0 \cup p^{-1}((\partial_\lambda X_w)^v)$$

(as sets), by the definition of $(\partial_\lambda X_w)^v$. In other words, $\pi^{-1}(E) \subseteq D$ as sets.

For (ii), observe that $\mathcal{I}_D = p^* L_\lambda(-Z)$ outside $\pi^{-1}(E)$. Thus, (ii) is equivalent to: $R^i \pi_*(p^* L_\lambda(-Z)) = 0$ outside $E$. We show that $R^i \pi_*(p^* L_\lambda(-Z)) = 0$ outside 0. Using the cartesian square above, it suffices to check that $R^i f_*(L_\lambda(-Z)) = 0$; by the Leray spectral sequence and the Serre vanishing theorem, this amounts to $H^i(X_w^v, L_{n\lambda}(-Z)) = 0$ for large $n$. But this holds by Proposition 1.

For (iii), recall that $X_w^v$ is split compatibly with $(\partial_\lambda X_w)^v \cup Z$. Let $\varphi$ be a compatible splitting; then $\varphi$ lifts uniquely to a splitting of $X$ compatibly with $X_0$ and with $p^{-1}((\partial_\lambda X_w)^v \cup Z)$. It follows that $X$ is split compatibly with $D$.

We thus obtain: $R^i \pi_*(\mathcal{I}_D) = 0$ for all $i \geq 1$. Since $Y$ is affine, this amounts to: $H^i(X, \mathcal{I}_D) = 0$ for all $i \geq 1$. On the other hand, since the morphism $p : X \longrightarrow X_w^v$ is affine, we obtain that $H^i(X, \mathcal{I}_D)$ is isomorphic to

$$H^i(X_w^v, p_* p^*(L_\lambda(-(\partial_\lambda X_w)^v) - Z)) = H^i(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v - Z) \otimes p_* \mathcal{O}_X).$$

Further, $p_* \mathcal{O}_X$ contains $\mathcal{O}_{X_w^v}$ as a direct factor. This yields

$$H^i(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v - Z)) = 0$$

for all $i \geq 1$. □

**Corollary 1.** *With the above notations, the restriction map*

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v)) \longrightarrow H^0(X_w^x, L_\lambda(-(\partial_\lambda X_w)^x))$$

*is surjective for any $x \in W^P$ such that $v \le x \le w$.*

*Proof.* We may reduce to the case that $\ell(x) = \ell(v) + 1$, that is, $X_w^x$ is an irreducible component of $(\partial X^v)_w$. Then $H^1(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v - X_w^x)) = 0$ by Proposition 2. Now the assertion follows from the exact sequence

$$0 \longrightarrow L_\lambda|_{X_w^v}(-(\partial_\lambda X_w)^v - X_w^x) \longrightarrow L_\lambda|_{X_w^v}(-(\partial_\lambda X_w)^v) \longrightarrow L_\lambda|_{X_w^x}(-(\partial_\lambda X_w)^x) \longrightarrow 0.$$

□

Notice finally that Lemma 5, Propositions 1 and 2, and Corollary 1 also hold in characteristic zero, as follows from the argument in [16] 3.7.

## 3. Filtrations

In this section, we shall obtain natural filtrations of the $T$–modules $H^0(X_w^v, L_\lambda)$ and $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$ (where $X_w^v$ is a Richardson variety in $G/P$, and $\lambda$ is a dominant character of $P$), and we shall describe their associated graded modules. For this, we shall construct a degeneration of $X_w^v$ embedded diagonally in $G/P \times G/P$, to a union of products of Richardson varieties.

Such a degeneration was obtained in [2] Theorem 16 for $X_w^v = G/B$, by using the wonderful compactification of the adjoint group of $G$; it was extended to certain subvarieties in $G/P$, including Schubert varieties, in [1] Theorem 2. Here we follow a direct, self–contained approach, at the cost of repeating some of the arguments in [2] and [1]. We begin by establishing a Künneth decomposition of the class of the diagonal of $G/P$, in the Grothendieck group of $G/P \times G/P$; such a decomposition is deduced in [2] from a degeneration of the diagonal.

Let $K(G/P \times G/P)$ be the Grothendieck group of the category of coherent sheaves on $G/P \times G/P$. The class of a coherent sheaf $\mathcal{F}$ in this group will be denoted by $[\mathcal{F}]$.

**Lemma 6.** *We have in $K(G/P \times G/P)$:*

$$[\mathcal{O}_{\mathrm{diag}(G/P)}] = [\mathcal{O}_{\bigcup_{x\in W^P} X_x \times X^x}] = \sum_{x\in W^P} [\mathcal{O}_{X_x}(-\partial X_x) \otimes \mathcal{O}_{X^x}] = \sum_{x\in W^P} [\mathcal{O}_{X_x} \otimes \mathcal{O}_{X^x}(-\partial X^x)].$$

*Proof.* Let $Z = \bigcup_{x\in W^P} X_x \times X^x$. We first claim that

$$[\mathcal{O}_Z] = \sum_{x\in W^P} [\mathcal{O}_{X_x}(-\partial X_x) \otimes \mathcal{O}_{X^x}].$$

Let $W^P = \{x_1, \dots, x_N\}$ be an indexing such that $i \le j$ whenever $x_i \le x_j$. Then one obtains easily:

$$(X_{x_i} \times X^{x_i}) \cap (\bigcup_{j<i} X_{x_j} \times X^{x_j}) = \partial X_{x_i} \times X^{x_i}.$$

Now let $\mathcal{O}_{Z,\ge i}$ be the subsheaf of $\mathcal{O}_Z$ consisting of those sections that vanish on $X_{x_j} \times X^{x_j}$ for each $j < i$. Then the $\mathcal{O}_{Z,\ge i}$ are a decreasing filtration of $\mathcal{O}_Z$, and

$$\mathcal{O}_{Z,\ge i}/\mathcal{O}_{Z,\ge i+1} \simeq \mathcal{O}_{Z,\ge i}|_{X_{x_i}\times X^{x_i}} \simeq \mathcal{O}_{X_i}(-\partial X_i) \otimes \mathcal{O}_{X^i}.$$

Further, $[\mathcal{O}_Z] = \sum_{i=1}^N [\mathcal{O}_{Z,\ge i}/\mathcal{O}_{Z,\ge i+1}]$ in $K(G/P \times G/P)$. This implies our claim.

One checks similarly that

$$[\mathcal{O}_Z] = \sum_{x \in W^P} [\mathcal{O}_{X_x} \otimes \mathcal{O}_{X^x}(-\partial X^x)],$$

using the increasing filtration of $\mathcal{O}_Z$ by the subsheaves $\mathcal{O}_{Z,\le i}$ consisting of those sections that vanish on $X_{x_j} \times X^{x_j}$ for each $j > i$.

To complete the proof, it suffices to check that

$$[\mathcal{O}_{\mathrm{diag}(G/P)}] = \sum_{x \in W^P} [\mathcal{O}_{X_x} \otimes \mathcal{O}_{X^x}(-\partial X^x)]. \tag{$*$}$$

For this, we recall some well–known facts on Grothendieck groups of flag varieties.

Since the Bruhat cells $C_x$, $x \in W^P$, form a cellular decomposition of $G/P$, the abelian group $K(G/P)$ is generated by the $[\mathcal{O}_{X_x}]$, $x \in W^P$. Likewise, it is generated by the $[\mathcal{O}_{X^y}]$, $y \in W^P$. Further, $K(G/P)$ is a ring for the product

$$[\mathcal{F}] \cdot [\mathcal{G}] = \sum_{i \ge 0} (-1)^i [Tor_i^{G/P}(\mathcal{F}, \mathcal{G})],$$

and the Euler characteristic of coherent sheaves yields an additive map

$$\begin{array}{rccc} \chi : & K(G/P) & \longrightarrow & \mathbb{Z} \\ & [\mathcal{F}] & \longmapsto & \chi(\mathcal{F}). \end{array}$$

Since $X_x$ and $X^y$ are Cohen–Macaulay and intersect properly in $G/P$, we have $Tor_i^{G/P}(\mathcal{O}_{X_x}, \mathcal{O}_{X^y}) = 0$ for all $i \ge 1$ (see [1] Lemma 1 for details). And since the intersection $X_x \cap X^y = X_x^y$ is reduced, we obtain

$$Tor_0^{G/P}(\mathcal{O}_{X_x}, \mathcal{O}_{X^y}) = \mathcal{O}_{X_x} \otimes_{\mathcal{O}_{G/P}} \mathcal{O}_{X^y} = \begin{cases} \mathcal{O}_{X_x^y} & \text{if } y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

Together with Proposition 1, it follows that

$$\chi([\mathcal{O}_{X_x}] \cdot [\mathcal{O}_{X^y}]) = \begin{cases} 1 & \text{if } y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we have in $K(G/P)$:

$$[\mathcal{O}_{X^y}] = \sum_{z\in W^P,\, z\geq y} [\mathcal{O}_{X^z}(-\partial X^z)]$$

(more generally, for any union $Z$ of opposite Schubert varieties, we have $[\mathcal{O}_Z] = \sum_{z\in W^P,\, X^z\subseteq Z}[\mathcal{O}_{X^z}(-\partial X^z)]$ by an easy induction, using the fact that intersections of unions of opposite Schubert varieties are reduced.) It follows that

$$\chi([\mathcal{O}_{X_x}]\cdot[\mathcal{O}_{X^y}(-\partial X^y)]) = \delta_{x,y}.$$

Thus, the $[\mathcal{O}_{X_x}]$, $x\in W^P$ form a basis for $K(G/P)$; further, the bilinear form $K(G/P)\times K(G/P)\longrightarrow \mathbb{Z}$, $(u,v)\longmapsto \chi(u\cdot v)$ is non–degenerate, and the dual basis of the $[\mathcal{O}_{X_x}]$ with respect to this pairing consists of the $[\mathcal{O}_{X^x}(-\partial X^x)]$.

It follows that a given class $u\in K(G/P\times G/P)$ is zero if and only if $\chi(u\cdot [\mathcal{O}_{X^y}(-\partial X^y)\otimes \mathcal{O}_{X_z}]) = 0$ for all $y,z\in W^P$. Further,

$$\chi([\mathcal{O}_{\mathrm{diag}(G/P)}]\cdot[\mathcal{O}_{X^y}(-\partial X^y)\otimes\mathcal{O}_{X_z}]) = \chi([\mathcal{O}_{X^y}(-\partial X^y)]\cdot[\mathcal{O}_{X_z}]) = \delta_{y,z},$$

whereas

$$\chi(\sum_{x\in W^P}[\mathcal{O}_{X_x}\otimes\mathcal{O}_{X^x}(-\partial X^x)]\cdot[\mathcal{O}_{X^y}(-\partial X^y)\otimes\mathcal{O}_{X_z}]) =$$

$$= \sum_{x\in W^P}\chi([\mathcal{O}_{X_x}]\cdot[\mathcal{O}_{X^y}(-\partial X^y)])\;\chi([\mathcal{O}_{X^x}(-\partial X^x)]\cdot[\mathcal{O}_{X_z}]) = \sum_{x\in W^P}\delta_{x,y}\delta_{x,z} = \delta_{y,z}.$$

This completes the proof of $(*)$, and hence of the lemma. □

We now construct a degeneration of the diagonal of any Richardson variety. Let $\theta:\mathbb{G}_m\longrightarrow T$ be a regular dominant one–parameter subgroup. Let $\mathcal{X}$ be the closure in $G/P\times G/P\times\mathbb{A}^1$ of the subset

$$\{(x,\theta(s)x,s)\mid x\in G/P,\ s\in k^*\}.$$

The variety $\mathcal{X}$ is invariant under the action of $\mathbb{G}_m\times T$ defined by

$$(s,t)(x,y,z) = (tx,\theta(s)ty,sz).$$

Consider the projections

$$p_1,p_2:\mathcal{X}\longrightarrow G/P,\ \pi:\mathcal{X}\longrightarrow\mathbb{A}^1.$$

Clearly, $\pi$ is proper and flat, and its fibers identify with closed subschemes of $G/P\times G/P$ via $p_1\times p_2$; this identifies the fiber at 1 with $diag(G/P)\simeq G/P$. By equivariance, every "general" fiber $\pi^{-1}(z)$, where $z\neq 0$, is also isomorphic to $G/P$.

We shall denote the "special" (scheme–theoretical) fiber $\pi^{-1}(0)$ by $F$, with projections

$$q_1,q_2:F\longrightarrow G/P.$$

Next let $v, w$ in $W^P$ such that $v \leq w$. Let $\mathcal{X}_w^v$ be the closure in $G/P \times G/P \times \mathbb{A}^1$ of the subset

$$\{(x, \theta(s)x, s) \mid x \in X_w^v,\ s \in k^*\}.$$

This is a subvariety of $\mathcal{X} \cap (X_w^v \times X_w^v \times \mathbb{A}^1)$, invariant under the action of $\mathbb{G}_m \times T$. We shall denote the restrictions of $p_1$, $p_2$, $\pi$ to $\mathcal{X}_w^v$ by the same letters; then $\pi$ is again proper and flat, and its "general" fibers are isomorphic to $X_w^v$. Let $F_w^v$ be the "special" fiber, with projections $q_1$, $q_2$ to $X_w^v$.

**Lemma 7.** (1) *The schemes $F$ and $F_w^v$ are reduced. Further,*

$$F = \bigcup_{x \in W^P} X_x \times X^x \text{ and } F_w^v = \bigcup_{x \in W^P,\, v \leq x \leq w} X_x^v \times X_w^x.$$

(2) *Choose a total ordering $\leq_t$ of $W^P$ such that $x \leq_t y$ whenever $x \leq y$. For $x \in W^P$, let $\mathcal{O}_{F,\leq_t x}$ (resp. $\mathcal{O}_{F,\geq_t x}$) be the subsheaf of $\mathcal{O}_F$ consisting of those sections that vanish identically on $X_y \times X^y$ for each $y >_t x$ (resp. $y <_t x$). Then the $\mathcal{O}_{F,\leq_t x}$ (resp. $\mathcal{O}_{F,\geq_t x}$) are an ascending (resp. descending) filtration of $\mathcal{O}_F$, with associated graded*

$$\bigoplus_{x \in W^P} \mathcal{O}_{X_x} \otimes \mathcal{O}_{X^x}(-\partial X^x), \text{ resp. } \bigoplus_{x \in W^P} \mathcal{O}_{X_x}(-\partial X_x) \otimes \mathcal{O}_{X^x}.$$

*The induced filtrations on the structure sheaf $\mathcal{O}_{F_w^v}$ have associated graded*

$$\bigoplus_{x \in W^P,\, v \leq x \leq w} \mathcal{O}_{X_x^v} \otimes \mathcal{O}_{X_w^x}(-(\partial X^x)_w), \text{ resp. } \bigoplus_{x \in W^P,\, v \leq x \leq w} \mathcal{O}_{X_x^v}(-(\partial X_x)^v) \otimes \mathcal{O}_{X_w^x}.$$

*The induced map*

$$\mathcal{O}_{F_w^v,\leq_t} \longrightarrow \operatorname{gr}_{\leq_t} \mathcal{O}_{F_w^v}$$

*is just the restriction to $X_x^v \times X_w^x$; the same holds for the induced map*

$$\mathcal{O}_{F_w^v,\geq_t} \longrightarrow \operatorname{gr}_{\geq_t} \mathcal{O}_{F_w^v}.$$

*Proof.* (1) Let $x \in W^P$. We claim that

$$C_x \times C^x \subseteq F.$$

To check this, consider the subset $xC^1$ of $G/P$. This is an open $T$–stable neighborhood of $e_x$ in $G/P$, isomorphic to affine space where $T$ acts linearly with weights the $\alpha \in x(R^- - R_P)$. Choose corresponding coordinate functions $z_\alpha$ on $xC^1$, then $C_x$ (resp. $C^x$) is the closed subset of $xC^1$ where $z_\alpha = 0$ whenever $\alpha \in R^-$ (resp. $\alpha \in R^+$). Let $z = (z_\alpha) \in xC^1$, then

$$\theta(s)z = (s^{\langle \alpha, \theta \rangle} z_\alpha).$$

Denote by $z_+$ (resp. $z_-$) the point of $C_x$ (resp. $C^x$) with coordinates $z_\alpha$, $\alpha \in R^+$ (resp. $\alpha \in R^-$). Let $z'(s)$ be the point of $xC^1$ with $\alpha$–coordinate $z_\alpha$ if $\alpha \in R^+$, and $\theta(s^{-1})z_\alpha$ otherwise. Since $\theta$ is regular dominant, we obtain

$$\lim_{s \to 0} (z'(s), \theta(s)z'(s), s) = (z_+, z_-, 0).$$

And since $z_+$ (resp. $z_-$) is an arbitrary point of $C_x$ (resp. $C^x$), this proves our claim.

The claim implies that $F$ contains $\bigcup_{x \in W^P} X_x \times X^x$ as a reduced closed subscheme. Let $\mathcal{I}$ be the ideal sheaf of this closed subscheme in $\mathcal{O}_F$; we regard $\mathcal{I}$ as a coherent sheaf on $G/P \times G/P$. Then we have in $K(G/P \times G/P)$:

$$[\mathcal{I}] = [\mathcal{O}_F] - [\mathcal{O}_{\bigcup_{x \in W^P} X_x \times X^x}] = [\mathcal{O}_{\mathrm{diag}(G/P)}] - [\mathcal{O}_{\bigcup_{x \in W^P} X_x \times X^x}] = 0,$$

where the first equality follows from the definition of $\mathcal{I}$, the second one from the fact that $\pi : \mathcal{X} \to \mathbb{A}^1$ is flat with fibers $F$ and $\mathrm{diag}(G/P)$, and the third one from Lemma 6. As a consequence, $\mathcal{I}$ is trivial (e.g., since its Hilbert polynomial is zero); this completes the proof for $F$.

In the case of $F_w^v$, notice that

$$F_w^v \subseteq F \cap (X^v \times X_w) = (\bigcup_{x \in W^P} X_x \times X^x) \cap (X^v \times X_w) = \bigcup_{x \in W^P, v \le x \le w} X_x^v \times X_w^x$$

as schemes, since all involved scheme–theoretic intersections are reduced. Further, we have in the Chow ring of $G/P \times G/P$:

$$\begin{aligned}[F_w^v] = [\mathrm{diag}(X_w^v)] = [\mathrm{diag}(G/P) \cap (X^v \times X_w)] &= [\mathrm{diag}(G/P)] \cdot [X^v \times X_w] \\ = [F] \cdot [X^v \times X_w] = [F \cap (X^v \times X_w)] &= \sum_{x \in W^P,\, v \le x \le w} [X_x^v \times X_w^x],\end{aligned}$$

since all involved intersections are proper and reduced. It follows that $F_w^v$ equals $\bigcup_{x \in W^P,\, v \le x \le w} X_x^v \times X_w^x$.

(2) has been established in the case of $F$, at the beginning of the proof of Lemma 6. The general case is similar. □

Next let $\lambda$ be a dominant character of $P$. This yields $T$–linearized line bundles $q_2^* L_\lambda$ on $F$ and on $F_w^v$, together with "adjunction" maps $H^0(G/P, L_\lambda) \longrightarrow H^0(F, q_2^* L_\lambda)$ and $H^0(X_w^v, L_\lambda) \longrightarrow H^0(F_w^v, q_2^* L_\lambda)$.

**Proposition 3.** (1) *These maps are isomorphisms, and the restriction map*

$$H^0(F, q_2^* L_\lambda) \longrightarrow H^0(F_w^v, q_2^* L_\lambda)$$

*is surjective.*

(2) *The ascending filtration of* $\mathcal{O}_F$ *yields an ascending filtration of the* $T$*–module* $H^0(F, q_2^* L_\lambda)$*, with associated graded*

$$\bigoplus_{x \in W^P} H^0(X^x, L_\lambda(-\partial X^x)).$$

(3) *The image of this filtration under restriction to* $F_w^v$ *has associated graded*

$$\bigoplus_{x \in W^P,\, v \le x \le w} H^0(X_w^x, L_\lambda(-(\partial X^x)_w)).$$

*Hence this is the associated graded of an ascending filtration*

$$H^0(X_w^v, L_\lambda)_{\le_t x}, \;\; v \le x \le w$$

*of* $H^0(X_w^v, L_\lambda)$*, compatible with the* $T$*–action and with restrictions to smaller Richardson varieties.*

(4) *The subspace*

$$H^0(X_w^v, L_\lambda)_{\le_t x} \subseteq H^0(X_w^v, L_\lambda)$$

*consists of those sections that vanish identically on* $X_w^y$ *for all* $y >_t x$*. Further, the map*

$$H^0(X_w^v, L_\lambda)_{\le_t x} \longrightarrow \operatorname{gr}_x H^0(X_w^v, L_\lambda) = H^0(X_w^x, L_\lambda(-(\partial X^x)_w))$$

*is just the restriction to* $X_w^x$.

*Proof.* (1) We have

$$H^0(F, q_2^* L_\lambda) = H^0(G/P, q_{2*} q_2^* L_\lambda) = H^0(G/P, L_\lambda \otimes q_{2*}\mathcal{O}_F)$$

by the projection formula. Further, the associated graded of the descending filtration of $\mathcal{O}_F$ is acyclic for $q_{2*}$; indeed, $H^i(X_x, \mathcal{O}_{X_x}(-\partial X_x)) = 0$ for all $i \ge 1$ and all $x \in W^P$, by Proposition 1. Notice also that $H^0(X_x, \mathcal{O}_{X_x}(-\partial X_x)) = 0$ for all $x \ne 1$, since $\partial X_x$ is a nonempty subscheme of the complete variety $X_x$. It follows that the natural map $\mathcal{O}_{G/P} \longrightarrow q_{2*}\mathcal{O}_F$ is an isomorphism. Hence the same holds for the map $H^0(G/P, L_\lambda) \longrightarrow H^0(F, q_2^* L_\lambda)$.

Likewise, the map $H^0(X_w^v, L_\lambda) \longrightarrow H^0(F_w^v, q_2^* L_\lambda)$ is an isomorphism as well. Since the restriction map $H^0(G/P, L_\lambda) \longrightarrow H^0(X_w^v, L_\lambda)$ is surjective by Proposition 1, the same holds for $H^0(F, q_2^* L_\lambda) \longrightarrow H^0(F_w^v, q_2^* L_\lambda)$.

(2) By Lemma 7 again, the ascending filtration of $\mathcal{O}_F$ yields one on $q_2^* L_\lambda$, with associated graded

$$\bigoplus_{x \in W^P} \mathcal{O}_{X_x} \otimes L_\lambda|_{X^x}(-\partial X^x).$$

The latter is acyclic by Proposition 1. It follows that $H^0(F, q_2^* L_\lambda)$ has an ascending filtration with associated graded as claimed.

(3) is checked similarly.

(4) We have

$$H^0(X_w^v, L_\lambda)_{\leq_t x} = H^0(F_w^v, q_2^* L_\lambda)_{\leq_t x} = H^0(F_w^v, q_2^* L_\lambda \otimes \mathcal{I}_{\cup_{y>_t x} X_y^v \times X_w^y})$$
$$= H^0(X_w^v, L_\lambda \otimes q_{2*}\mathcal{I}_{\cup_{y>_t x} X_y^v \times X_w^y})$$

by the projection formula. Further, $q_{2*}\mathcal{O}_{F_w^v} = \mathcal{O}_{X_w^v}$ as seen in the proof of (1). It follows that

$$q_{2*}\mathcal{I}_{\cup_{y>_t x} X_y^v \times X_w^y} = \mathcal{I}_{\cup_{y>_t x} X_w^y}.$$

This implies our statement. □

We now construct a similar filtration of the $T$–submodule

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v)) \subseteq H^0(X_w^v, L_\lambda).$$

For this, we define a sheaf on $F_w^v$ by

$$q_2^* L_\lambda(-(\partial_\lambda X_w)^v) = (q_2^* L_\lambda) \otimes_{\mathcal{O}_{F_w^v}} \mathcal{I}_{q_2^{-1}((\partial_\lambda X_w)^v)}.$$

This is a subsheaf of $q_2^* L_\lambda$; it may differ from the pullback sheaf of $L_\lambda(-(\partial_\lambda X_w)^v)$ under $q_2$. We also have an "adjunction" map

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v)) \longrightarrow H^0(F_w^v, q_2^* L_\lambda(-(\partial_\lambda X_w)^v)).$$

In particular, we obtain a map

$$H^0(X_w, L_\lambda(-\partial_\lambda X_w)) \longrightarrow H^0(F_w, q_2^* L_\lambda(-\partial_\lambda X_w)).$$

**Proposition 4.** (1) *These maps are isomorphisms, and the restriction map*

$$H^0(F_w, q_2^* L_\lambda(-\partial_\lambda X_w)) \longrightarrow H^0(F_w^v, q_2^* L_\lambda(-(\partial_\lambda X_w)^v))$$

*is surjective.*

(2) *The ascending filtration of $\mathcal{O}_{F_w}$ yields an ascending filtration of the $T$–module $H^0(F_w, q_2^* L_\lambda(-\partial_\lambda X_w))$, with associated graded*

$$\bigoplus_{x \in W^P,\, x \leq w} H^0(X_w^x, L_\lambda(-(\partial_\lambda X_w)^x - (\partial X^x)_w)).$$

(3) *The image of this filtration under restriction to $F_w^v$ has associated graded*

$$\bigoplus_{x \in W^P,\, v \leq x \leq w} H^0(X_w^x, L_\lambda(-(\partial_\lambda X_w)^x - (\partial X^x)_w)).$$

*Hence this is also the associated graded of an ascending filtration*

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))_{\leq_t x}, \ v \leq x \leq w$$

*of $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$, compatible with the $T$–action and with restrictions to smaller Richardson varieties.*

(4) *The subspace*

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))_{\leq_t x} \subseteq H^0(X_w^v, L_\lambda)$$

*consists of those sections that vanish identically on* $(\partial_\lambda X_w)^v$ *and on* $X_w^y$ *for all* $y >_t x$*. Further, the map*

$$H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))_{\leq_t x} \longrightarrow \mathrm{gr}_x\, H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v)) = \\ = H^0(X_w^x, L_\lambda(-(\partial_\lambda X_w)^x - (\partial X^x)_w))$$

*is just the restriction to* $X_w^x$*.*

*Proof.* It follows from Lemma 7 that the sheaf $q_2^* L_\lambda(-(\partial_\lambda X_w)^v)$ on $F_w^v$ admits a descending filtration with associated graded

$$\bigoplus_{x \in W^P, v \leq x \leq w} \mathcal{O}_{X_x^v}(-(\partial X_x)^v) \otimes L_\lambda|_{X_w^x}(-(\partial_\lambda X_w)^v),$$

and an ascending filtration with associated graded

$$\bigoplus_{x \in W^P, v \leq x \leq w} \mathcal{O}_{X_x^v} \otimes L_\lambda|_{X_w^x}(-(\partial_\lambda X_w)^v - (\partial X_x)^v).$$

As in the proof of Proposition 3, the associated graded of the first filtration is acyclic for $q_{2*}$; it follows that the adjunction map is an isomorphism. Further, the restriction map

$$H^0(X_w, L_\lambda(-\partial_\lambda X_w)) \longrightarrow H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$$

is surjective, by Corollary 1. Finally, the associated graded of the second filtration is acyclic, by Proposition 2. These facts imply our statements, as in the proof of Proposition 3. □

**Remarks.**

(1) By Proposition 3, the $H^0(G/P, L_\lambda)_{\leq_t x}$ are $B^-$–submodules of $H^0(G/P, L_\lambda)$. Likewise, the descending filtration of $\mathcal{O}_F$ yields a descending filtration of $H^0(G/P, L_\lambda)$ by $B$–submodules $H^0(G/P, L_\lambda)_{\geq_t x}$, consisting of those sections that vanish on $X_y$ whenever $y <_t x$.
(2) We may have defined directly the preceding filtrations by Propositions 3 (4) and 4 (4), without using the degeneration of the diagonal constructed in Lemma 7. In fact, this alternative definition suffices for the construction of a standard basis in the next section. But the degeneration of the diagonal will play an essential rôle in the section on standard products.

## 4. Construction of a standard basis

In this section, we fix a dominant weight $\lambda$ and we consider Richardson varieties in $G/P$, where $P = P_\lambda$. We shall construct a basis of $H^0(G/P, L_\lambda)$ adapted to the filtrations of Propositions 3 and 4. We first prove the key

**Lemma 8.** *Let $v \le w \in W^\lambda$. Then any element of $H^0(X_w^v, L_\lambda(-(\partial X_w)^v - (\partial X^v)_w))$ can be lifted to an element of $H^0(G/P, L_\lambda)$ that vanishes identically on all Schubert varieties $X_y$, $y \not\geq w$, and on all opposite Schubert varieties $X^x$, $x \not\leq v$.*

*Proof.* Put

$$X = X_w^v \text{ and } Y = (\bigcup_{y \not\geq w} X_y) \cup (\bigcup_{x \not\leq v} X^x).$$

Notice that

$$X \cap Y = (\partial X_w)^v \cup (\partial X^v)_w$$

(as schemes), since any intersection of unions of Richardson varieties is reduced. This yields an exact sequence

$$0 \longrightarrow \mathcal{I}_{X \cup Y} \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{I}_Y \otimes_{\mathcal{O}_{G/P}} \mathcal{O}_X \simeq \mathcal{O}_X(-X \cap Y) \longrightarrow 0.$$

Tensoring by $L_\lambda$ and taking the associated long exact sequence of cohomology groups yields an exact sequence

$$H^0(G/P, L_\lambda \otimes \mathcal{I}_Y) \longrightarrow H^0(X, L_\lambda(-X \cap Y)) \longrightarrow H^1(X, L_\lambda \otimes \mathcal{I}_{X \cup Y}).$$

Further, $H^1(X, L_\lambda \otimes \mathcal{I}_{X \cup Y}) = 0$ by Lemma 5; this completes the proof. □

**Definition 4.** *For any $v \le w \in W^\lambda$, let*

$$H_w^v(\lambda) = H^0(X_w^v, L_\lambda(-(\partial X_w)^v - (\partial X^v)_w))$$

*and*

$$\chi_w^v(\lambda) = \{\textit{the weights of the } T--\textit{module } H_w^v(\lambda)\},$$

*these weights being counted with multiplicity. Let*

$$\{p_{w,v}^\xi,\ \xi \in \chi_w^v(\lambda)\}$$

*be a basis for $H_w^v(\lambda)$, where each $p_{w,v}^\xi$ is a $T$–eigenvector of weight $\xi$.*

*For any triple $(w, v, \xi)$ as above, let $p_\pi$ be a lift of $p_{w,v}^\xi$ in $H^0(G/P, L_\lambda)$ such that:*
*$p_\pi$ is a $T$–eigenvector of weight $\xi$, and*
*$p_\pi$ vanishes identically on all $X_y$, $y \not\geq w$ and on all $X^x$, $x \not\leq v$.*

(The existence of such lifts follows from Lemma 8.) If $v = w$, then $X_w^v$ consists of the point $e_w$, and hence $\chi_w^v(\lambda)$ consists of the weight $-w(\lambda)$. We then denote the unique $p_{w,v}^\xi$ by $p_w$. Its lift to $H^0(G/P, L_\lambda)$ is unique; it is the extremal weight vector $p_w$.

**Definition 5.** *Let $\pi = (w, v, \xi)$ be as in Definition 4. We set $i(\pi) = w$, $e(\pi) = v$, and call them respectively the initial and end elements of $\pi$.*

By construction of the $p_\pi$ and Lemma 1, we obtain:

**Lemma 9.** *With notations as above, we have for $x, y \in W^\lambda$:*

$$p_\pi|_{X_y^x} \neq 0 \Longleftrightarrow X_{i(\pi)}^{e(\pi)} \subseteq X_y^x \Longleftrightarrow x \leq e(\pi) \leq i(\pi) \leq y.$$

**Proposition 5.** *The restrictions to $X_v^w$ of the $p_\pi$ where $i(\pi) = w$, $e(\pi) \geq v$ form a basis for the $T$–module $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))$, adapted to its ascending filtration $\leq_t$ of Proposition 4.*

*Proof.* By construction, $p_\pi$ vanishes identically on $X^x$ for any $x \not\leq e(\pi)$, and hence for any $x >_t e(\pi)$. Thus, $p_\pi \in H^0(G/P, L_\lambda)_{\leq_t e(\pi)}$ by Proposition 3. Further, the image of $p_\pi$ in the associated graded is just its restriction to $X^{e(\pi)}$.

Together with Lemma 9, it follows that the restrictions of the $p_\pi$ to $X_w^v$ belong to $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))_{\leq_t e(\pi)}$, and that their images in the associated graded $H_w^{e(\pi)}(\lambda)$ are the restrictions of the $p_\pi$ to $X_w^{e(\pi)}$; by construction, these images form a basis of $H_w^{e(\pi)}(\lambda)$. □

Now the $T$–module $H^0(X_w^v, L_\lambda)$ has a descending filtration by the submodules

$$H^0(X_w^v, L_\lambda(-(\partial X_w)^v))_{\geq_t x}$$

consisting of those sections that vanish identically on $X_y^v$ whenever $y <_t x$. And like in Proposition 3, the associated graded is

$$\bigoplus_{x \in W^P, v \leq x \leq w} H^0(X_x^v, L_\lambda(-(\partial X_x)^v)).$$

Further, we may check as in the proof of Proposition 5 that

$$p_\pi|_{X_w^v} \in H^0(X_w^v, L_\lambda(-(\partial X_w)^v))_{\geq_t i(\pi)}$$

whenever $i(\pi) \geq w$, and the image of $p_\pi$ in the associated graded is just its restriction to $X_{i(\pi)}^v$. Together with Proposition 5, this implies

**Proposition 6.** *The restrictions to $X_w^v$ of the $p_\pi$ where $v \leq e(\pi) \leq i(\pi) \leq w$ form a basis of $H^0(X_w^v, L_\lambda)$; the $p_\pi$ where $v \not\leq e(\pi)$ or $i(\pi) \not\leq w$ form a basis of the kernel of the restriction map $H^0(G/P, L_\lambda) \longrightarrow H^0(X_w^v, L_\lambda)$.*

In view of Proposition 6, the restriction to $p_\pi$ to $X_y^x$, where $x \leq e(\pi) \leq i(\pi) \leq y$, will be denoted by just $p_\pi$.

**Definition 6.** *Set*

$$\Pi(\lambda) = \{(v, w, \xi) \mid v, w \in W^\lambda,\ v \leq w,\ \xi \in \chi_w^v(\lambda)\}.$$

*For any $v, w \in W^\lambda, v \leq w$, set*

$$\Pi_w^v(\lambda) := \{\pi \mid v \leq e(\pi) \leq i(\pi) \leq w\}.$$

*In view of Lemma 9, we have,* $\Pi_w^v(\lambda) = \{\pi \in \Pi(\lambda) \mid p_\pi|_{X_w^v} \neq 0\}$.

*More generally, for a union $Z$ of Richardson varieties, define*

$$\Pi_Z(\lambda) = \{\pi \in \Pi(\lambda) \mid p_\pi|_Z \neq 0\}.$$

**Theorem 1.** *Let $Z$ be a union of Richardson varieties. Then $\{p_\pi|_Z, \pi \in \Pi_Z(\lambda)\}$ is a basis for $H^0(Z, L_\lambda)$, and $\{p_\pi, \pi \in \Pi(\lambda) - \Pi_Z(\lambda)\}$ is a basis for the kernel of the restriction map $H^0(G/P, L_\lambda) \longrightarrow H^0(Z, L_\lambda)$.*

*Proof.* By definition of $\Pi_Z(\lambda)$, it suffices to prove the first assertion. Let $Z = \cup_{i=1}^r X_{w_i}^{v_i}$. We shall prove the result by induction on $r$ and $\dim Z$. Write $Z = X \cup Y$ where $X = X_{w_i}^{v_i}$ for some $i$, and $\dim X = \dim Z$. Then $X \cap Y$ is a union of Richardson varieties of dimension $< \dim Z$. Consider the exact sequence

$$(*) \qquad 0 \to \mathcal{O}_Z = \mathcal{O}_{X\cup Y} \to \mathcal{O}_X \oplus \mathcal{O}_Y \to \mathcal{O}_{X\cap Y} \to 0.$$

Tensoring by $L_\lambda$, taking global sections and using the vanishing of $H^1(Z, L_\lambda)$ (Lemma 5), we obtain the exact sequence

$$0 \longrightarrow H^0(Z, L_\lambda) \longrightarrow H^0(X, L_\lambda) \oplus H^0(Y, L_\lambda) \longrightarrow H^0(X \cap Y, L_\lambda) \longrightarrow 0.$$

In particular, denoting $\dim H^0(Z, L_\lambda)$ by $h^0(Z, L_\lambda)$ etc., we obtain,

$$h^0(Z, L_\lambda) = h^0(X, L_\lambda) + h^0(Y, L_\lambda) - h^0(X \cap Y, L_\lambda).$$

We have by hypothesis (and induction hypothesis), $h^0(X, L_\lambda) = \#\Pi_X(\lambda)$, $h^0(Y, L_\lambda) = \#\Pi_Y(\lambda)$, $h^0(X \cap Y, L_\lambda) = \#\Pi_{X\cap Y}(\lambda)$. Thus we obtain,

$$(1) \qquad h^0(Z, L_\lambda) = \#\Pi_X(\lambda) + \#\Pi_Y(\lambda) - \#\Pi_{X\cap Y}(\lambda).$$

On the other hand we have,

$$(2) \qquad \Pi_Z(\lambda) = (\Pi_X(\lambda) \,\dot\cup\, \Pi_Y(\lambda)) \setminus \Pi_{X\cap Y}(\lambda).$$

¿From (1) and (2), we obtain, $h^0(Z, L_\lambda) = \#\Pi_Z(\lambda)$. Further, the $p_\pi|_Z$, $\pi \in \Pi_Z(\lambda)$, span $H^0(Z, L_\lambda)$ (since the $p_\pi$, $\pi \in \Pi(\lambda)$, span $H^0(G/P, L_\lambda)$, and the restriction map $H^0(G/P, L_\lambda) \to H^0(Z, L_\lambda)$ is surjective). Thus, the $p_\pi|_Z$, $\pi \in \Pi_Z(\lambda)$, are a basis of $H^0(Z, L_\lambda)$. □

## 5. Standard monomials

Let $\lambda$, $\mu$ be dominant weights such that $P_\lambda = P_\mu := P$. Consider the product map

$$H^0(G/P, L_\lambda) \otimes H^0(G/P, L_\mu) \longrightarrow H^0(G/P, L_{\lambda+\mu}).$$

This map is surjective by [16] 2.2 and 3.5. Using Proposition 1, it follows that the product map

$$H^0(X_w^v, L_\lambda) \otimes H^0(X_w^v, L_\mu) \longrightarrow H^0(X_w^v, L_{\lambda+\mu})$$

is also surjective, for any $v \le w$ in $W^P$. We shall construct a basis for $H^0(X_w^v, L_{\lambda+\mu})$ from the bases of $H^0(X_w^v, L_\lambda)$, $H^0(X_w^v, L_\mu)$ obtained in Theorem 1. For this, we need the following

**Definition 7.** *Let $v, w \in W^P, v \le w$. Let $\varphi \in \Pi_w^v(\lambda)$ and $\psi \in \Pi_w^v(\mu)$. The pair $(\varphi, \psi)$ is called standard on $X_w^v$ if*

$$v \le e(\psi) \le i(\psi) \le e(\varphi) \le i(\varphi) \le w.$$

*Then the product $p_\varphi p_\psi \in H^0(G/P, L_{\lambda+\mu})$ is called standard on $X_w^v$ as well.*

Clearly, we have

**Lemma 10.** *Let $p_\varphi p_\psi$ be a standard product on $G/P$ and let $v \le w \in W^P$. Then*

$$p_\varphi p_\psi|_{X_w^v} \ne 0 \Longleftrightarrow v \le e(\psi) \le i(\varphi) \le w.$$

**Proposition 7.** *The standard products on $X_w^v$ form a basis of $H^0(X_w^v, L_{\lambda+\mu})$. The standard products on $G/P$ that are not standard on $X_w^v$ form a basis of the kernel of the restriction map $H^0(G/P, L_{\lambda+\mu}) \longrightarrow H^0(X_w^v, L_{\lambda+\mu})$.*

*Proof.* Consider the $T$–linearized invertible sheaf $q_1^* L_\mu \otimes q_2^* L_\lambda$ on $F_w^v$. By Lemma 7, the ascending filtration of $\mathcal{O}_{F_w^v}$ yields one of that sheaf, with associated graded

$$\bigoplus_{x \in W^P, v \le x \le w} L_\mu|_{X_x^v}(-(\partial X_x)^v) \otimes L_\lambda|_{X_w^x}.$$

By Proposition 1, the latter sheaf is acyclic. This yields an ascending filtration of the $T$–module $H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$, with associated graded

$$\bigoplus_{x \in W^P, v \le x \le w} H^0(X_x^v, L_\mu(-(\partial X_x)^v)) \otimes H^0(X_w^x, L_\lambda);$$

it also follows that $H^i(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda) = 0$ for all $i \ge 1$.

By Proposition 3, we may identify $H^0(X_w^v, L_\lambda)$ with $H^0(F_w^v, q_2^* L_\lambda)$; likewise, we may identify $H^0(X_w^v, L_\mu)$ with $H^0(F_w^v, q_1^* L_\mu)$. Using the multiplication map

$$H^0(F_w^v, q_1^* L_\mu) \otimes H^0(F_w^v, q_2^* L_\lambda) \longrightarrow H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda),$$

this defines "dot products" in $H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$.

Let $x \in W^P$ such that $v \le x \le w$. Recall that the $p_\psi$, $v \le e(\psi) \le i(\psi) = x$, are a basis of $H^0(X_x^v, L_\mu(-(\partial X_x)^v))$. Further, the $p_\varphi$, $x \le e(\varphi) \le i(\varphi) \le w$, are a basis of $H^0(X_w^x, L_\lambda)$. Thus, the dot products $p_\psi \cdot p_\varphi$, where there exists $x \in W^P$ such that

$$v \le e(\psi) \le i(\psi) = x \text{ and } x \le e(\varphi) \le i(\varphi) \le w,$$

restrict to a basis of $H^0(X_x^v, L_\mu(-(\partial X_x)^v)) \otimes H^0(X_w^x, L_\lambda)$. By construction of the filtration of $H^0(F_w^v, q_1^* L_\mu \otimes q_2^* L_\lambda)$, it follows that the standard dot products are a basis of that space.

Consider now the $T$–linearized invertible sheaf $p_1^*L_\mu \otimes p_2^*L_\lambda$ on $\mathcal{X}_w^v$. This sheaf is flat on $\mathbb{A}^1$; by vanishing of $H^1(F_w^v, q_1^*L_\mu \otimes q_2^*L_\lambda)$ and semicontinuity, it follows that the restriction

$$H^0(\mathcal{X}_w^v, p_1^*L_\mu \otimes p_2^*L_\lambda) \longrightarrow H^0(F_w^v, q_1^*L_\mu \otimes q_2^*L_\lambda)$$

is surjective, and that $H^0(\mathcal{X}_w^v, p_1^*L_\mu \otimes p_2^*L_\lambda)$ is a free module over $H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}_1}) = k[z]$, generated by any lift of its quotient space $H^0(F_w^v, q_1^*L_\mu \otimes q_2^*L_\lambda)$.

We now construct such a lift, as follows. Consider the adjunction maps

$$H^0(X_w^v, L_\lambda) \longrightarrow H^0(\mathcal{X}_w^v, p_2^*L_\lambda) \text{ and } H^0(X_w^v, L_\mu) \longrightarrow H^0(\mathcal{X}_w^v, p_1^*L_\mu).$$

These yield dot products $p_\psi \cdot p_\varphi$ in $H^0(\mathcal{X}_w^v, p_1^*L_\mu \otimes p_2^*L_\lambda)$ which lift the corresponding products in $H^0(F_w^v, q_1^*L_\mu \otimes q_2^*L_\lambda)$. Since the latter standard products are a basis of that space, the standard dot products $p_\psi \cdot p_\varphi$ are a basis of $H^0(\mathcal{X}_w^v, p_1^*L_\mu \otimes p_2^*L_\lambda)$ over $k[z]$. Therefore, they restrict to a basis of the space of sections of $p_1^*L_\mu \otimes p_2^*L_\lambda$ over any fiber of $\pi$. But the fiber at 1 is $diag(X_w^v)$, and the restriction of $p_1^*L_\mu \otimes p_2^*L_\lambda$ to that fiber is just $L_{\lambda+\mu}$ whereas the restrictions of the dot products are just the usual products. We have proved that the standard products on $X_w^v$ form a basis of $H^0(X_w^v, L_{\lambda+\mu})$.

To complete the proof, notice that any standard product on $G/P$ that is not standard on $X_w^v$ vanishes identically on that subvariety, by Lemma 10. □

**Remark.** The proof of Proposition 7 relies on the fact that the special fiber $F_w^v$ of the flat family $\pi : \mathcal{X}_w^v \to \mathbb{A}^1$ equals $\bigcup_{x \in W^P, v \le x \le w} X_x^v \times X_w^x$. Conversely, this fact can be recovered from Proposition 7, as follows.

We have the equalities of Euler characteristics:

$$\chi(F, q_1^*L_\mu \otimes q_2^*L_\lambda) = \chi(G/P, L_{\lambda+\mu}) = \sum_{x \in W^P} \chi(X_x, L_\mu(-\partial X_x))\, \chi(X^x, L_\lambda),$$

where the first equality holds by flatness of $\pi$, and the second one by Propositions 5, 6 and 7. It follows that

$$\chi(F, q_1^*L_\mu \otimes q_2^*L_\lambda) = \chi(\bigcup_{x \in W^P} X_x \times X^x, q_1^*L_\mu \otimes q_2^*L_\lambda).$$

Since $F$ contains $\bigcup_{x \in W^P} X_x \times X^x$ by the first claim in the proof of Lemma 7, and $\lambda$, $\mu$ are arbitrary dominant $P$–regular weights, it follows that $F = \bigcup_{x \in W^P} X_x \times X^x$ (e.g., since both have the same Hilbert polynomial). Now the argument of Lemma 7 yields $F_w^v = \bigcup_{x \in W^P, v \le x \le w} X_x^v \times X_w^x$.

We now extend Proposition 7 to unions of Richardson varieties.

**Definition 8.** *Let $\Pi(\lambda, \mu)$ be the set of all standard pairs $(\varphi, \psi)$ where $\varphi \in \Pi(\lambda)$ and $\psi \in \Pi(\mu)$. For $v \le w \in W^P$, let $\Pi_w^v(\lambda, \mu)$ be the subset of standard pairs on $X_w^v$. In* *view of Lemma 10, we have*

$$\Pi_w^v(\lambda,\mu)=\{(\varphi,\psi)\in\Pi(\lambda,\mu)\mid p_\varphi p_\psi|_{X_w^v}\neq 0\}.$$

*Finally, for a union $Z$ of Richardson varieties, let*

$$\Pi_Z(\lambda,\mu)=\{(\varphi,\psi)\in\Pi(\lambda,\mu)\mid p_\varphi p_\psi|_Z\neq 0\}.$$

Now arguing as in the proof of Theorem 1, we obtain

**Theorem 2.** *Let $Z$ be a union of Richardson varieties in $G/P$. Then the products $p_\varphi p_\psi$, where $(\varphi,\psi)\in\Pi_Z(\lambda,\mu)$, form a basis of $H^0(Z,L_{\lambda+\mu})$. The products $p_\varphi p_\psi$, where $(\varphi,\psi)\in\Pi(\lambda,\mu)-\Pi_Z(\lambda,\mu)$, form a basis of the kernel of the restriction map $H^0(G/P,L_{\lambda+\mu})\longrightarrow H^0(Z,L_{\lambda+\mu})$.*

**Corollary 2.** *For any $\varphi\in\Pi(\lambda)$ and $\psi\in\Pi(\mu)$, the product $p_\varphi p_\psi\in H^0(G/P,L_{\lambda+\mu})$ is a linear combination of standard products $p_{\varphi'}p_{\psi'}$ where $i(\varphi')\geq i(\varphi)$ and $e(\psi')\leq e(\psi)$.*

*Proof.* Notice that $p_\varphi p_\psi$ vanishes identically on all $X_y$ where $y\not\geq i(\varphi)$, and on all $X^x$ where $x\not\leq e(\psi)$. By Theorem 2, it follows that $p_\varphi p_\psi$ is a linear combination of standard products $p_{\varphi'}p_{\psi'}$, where $i(\varphi)'\not\leq y$ whenever $y\not\geq i(\varphi)$, and $e(\psi)'\not\geq x$ whenever $x\not\leq e(\psi)$. But this means exactly that $i(\varphi')\geq i(\varphi)$ and $e(\psi')\leq e(\psi)$. □

Next we consider a family of dominant weights $\lambda_1,\ldots,\lambda_m$ such that $P=P_{\lambda_1}=\cdots=P_{\lambda_m}$. For any union $Z$ of Richardson varieties in $G/P$, we shall construct a basis of $H^0(Z,L_{\lambda_1+\cdots+\lambda_m})$, in terms of *standard monomials of degree $m$*. These are defined as follows.

**Definition 9.** *Let $\pi_i\in\Pi(\lambda_i)$ for $1\leq i\leq m$. Then the sequence $\underline{\pi}:=(\pi_1,\pi_2,\ldots,\pi_m)$ is standard if*

$$e(\pi_m)\leq i(\pi_m)\leq\cdots\leq e(\pi_1)\leq i(\pi_1).$$

*Further, let $v,w\in W^P$ such that $v\leq w$; then $\underline{\pi}$ is standard on $X_w^v$ if*

$$v\leq e(\pi_m)\leq i(\pi_m)\leq\cdots\leq e(\pi_1)\leq i(\pi_1)\leq w.$$

*Finally, $\underline{\pi}$ is standard on $Z=\cup X_{w_i}^{v_i}$ if it is standard on $X_{w_i}^{v_i}$ for some $i$.*

Set

$$\Pi_w^v(\lambda_1,\ldots,\lambda_m)=\{\underline{\pi}=(\pi_1,\pi_2,\ldots,\pi_m)\,|\,\underline{\pi}\text{ is standard on }X_w^v\},$$
$$\Pi_Z(\lambda_1,\ldots,\lambda_m)=\{\underline{\pi}=(\pi_1,\pi_2,\ldots,\pi_m)\,|\,\underline{\pi}\text{ is standard on }Z\}.$$

**Definition 10.** *Given $\underline{\pi}=(\pi_1,\pi_2,\ldots,\pi_m)$, set $p_{\underline{\pi}}:=p_{\pi_1}\cdots p_{\pi_m}$.*
*Note that $p_{\underline{\pi}}\in H^0(G/P,L_{\lambda_1+\cdots+\lambda_m})$. If $\underline{\pi}$ is standard, then we call $p_{\underline{\pi}}$ a standard monomial on $G/P$. If $\underline{\pi}$ is standard on $X_w^v$ (resp. $Z$) , then we call $p_{\underline{\pi}}$ a standard monomial on $X_w^v$ (resp. $Z$).*

By Theorem 2 and induction on $m$, we obtain

**Corollary 3.** *Let $Z$ be a union of Richardson varieties in $G/P$ and let $\lambda_1, \dots, \lambda_m$ be dominant weights such that $P = P_{\lambda_1} = \cdots = P_{\lambda_m}$. Then the monomials $p_{\underline{\pi}}$ where $\underline{\pi} \in \Pi_Z(\lambda_1, \dots, \lambda_m)$ form a basis of $H^0(Z, L_{\lambda_1+\cdots+\lambda_m})$. Further, the monomials $p_{\underline{\pi}}$ where $\underline{\pi} \in \Pi(\lambda_1, \dots, \lambda_m) - \Pi_Z(\lambda_1, \dots, \lambda_m)$, form a basis of the kernel of the restriction map $H^0(G/P, L_{\lambda_1+\cdots+\lambda_m}) \longrightarrow H^0(Z, L_{\lambda_1+\cdots+\lambda_m})$.*

As an application, we determine the equations of unions of Richardson varieties in their projective embeddings given by very ample line bundles on $G/P$. Let $\lambda$ be a dominant $P$–regular weight. For any $\pi_1, \pi_2 \in \Pi(\lambda)$, we have in $H^0(G/P, L_{2\lambda})$:

$$p_{\pi_1} p_{\pi_2} - \sum a_{\pi_1', \pi_2'} p_{\pi_1'} p_{\pi_2'} = 0,$$

where $a_{\pi_1', \pi_2'} \in k$ and the sum is over those standard pairs $(\pi_1', \pi_2') \in \Pi(\lambda, \lambda)$ such that $i(\pi_1') \geq i(\pi_1)$ and $e(\pi_2') \leq e(\pi_2)$ (as follows from Corollary 2).

**Definition 11.** *The preceding elements $p_{\pi_1} p_{\pi_2} - \sum a_{\pi_1', \pi_2'} p_{\pi_1'} p_{\pi_2'}$ when regarded in $S^2 H^0(G/P, L_\lambda)$, will be called the quadratic straightening relations.*

**Corollary 4.** *Let $\lambda$ be a regular dominant character of $P$.*

(1) *The multiplication map*

$$\bigoplus_{m=0}^{\infty} S^m H^0(G/P, L_\lambda) \longrightarrow \bigoplus_{m=0}^{\infty} H^0(G/P, L_{m\lambda})$$

*is surjective, and its kernel is generated as an ideal by the quadratic straightening relations.*

(2) *For any union $Z$ of Richardson varieties in $G/P$, the restriction map*

$$\bigoplus_{m=0}^{\infty} H^0(G/P, L_{m\lambda}) \longrightarrow \bigoplus_{m=0}^{\infty} H^0(Z, L_{m\lambda})$$

*is surjective. Its kernel is generated as an ideal by the $p_\pi$, $\pi \in \Pi(\lambda) - \Pi_Z(\lambda)$ together with the standard products $p_{\pi_1} p_{\pi_2}$ where $i(\pi_1) \not\leq w$ or $e(\pi_2) \not\geq v$ whenever $X_w^v$ is an irreducible component of $Z$. If in addition $Z$ is a union of Richardson varieties $X_w^v$ all having the same $w$, then the $p_\pi$, $\pi \in \Pi(\lambda) - \Pi_Z(\lambda)$ suffice.*

*Proof.* (1) By [16] Theorem 3.11, the multiplication map is surjective, and its kernel is generated as an ideal by the kernel $K$ of the map $S^2 H^0(G/P, L_\lambda) \longrightarrow H^0(G/P, L_{2\lambda})$. Let $J$ be the subspace of $S^2 H^0(G/P, L_\lambda)$ generated by all quadratic straightening relations. Then $J \subseteq K$, and the quotient space $S^2 H^0(G/P, L_\lambda)/J$ is spanned by the images of the standard products. Further, their images in $S^2 H^0(G/P, L_\lambda)/K \simeq H^0(G/P, L_{2\lambda})$ form a basis, by Proposition 7. It follows that $J = K$.

(2) The first assertion follows from Lemma 5. Consider a standard monomial $p_{\underline{\pi}} = p_{\pi_1} \cdots p_{\pi_m} \in H^0(G/P, L_{m\lambda})$. By Corollary 3, $p_{\underline{\pi}}$ vanishes identically on $Z$ if and

only if: $i(\pi_1) \not\leq w$ or $e(\pi_m) \not\geq v$ for all irreducible components $X_w^v$. This amounts to: $p_{\pi_1}p_{\pi_m}$ vanishes identically on $Z$. If in addition $w$ is independent of the component, then $p_{\pi_1}$ or $p_{\pi_m}$ vanishes identically on $Z$; further, $p_{\pi_1}p_{\pi_m}$ is a standard product on $G/P$. This implies the remaining assertions, since the kernel of $H^0(G/P, L_{m\lambda}) \longrightarrow H^0(Z, L_{m\lambda})$ is spanned by those standard monomials on $G/P$ that are not standard on $Z$ (Corollary 3). □

**Remark.** In particular, the $p_\pi$, where $\pi \in \Pi(\lambda) - \Pi_Z(\lambda)$, generate the homogeneous ideal of $Z$ in $G/P$, whenever $Z$ is a union of Schubert varieties (or a union of opposite Schubert varieties). But this does not extend to arbitrary unions of Richardson varieties, as shown by the obvious example where $G/P = \mathbb{P}^1$, $Z = \{0, \infty\}$ and $L_\lambda = \mathcal{O}(1)$; then $\Pi(\lambda) = \Pi_Z(\lambda)$.

## 6. Weights of classical type

In this section, we shall determine the "building blocks"

$$H_w^v(\lambda) = H^0(X_w^v, L_\lambda(-(\partial X_w)^v - (\partial X^v)_w))$$

in the case where the dominant weight $\lambda$ is of classical type (as introduced in [9], cf. the next definition). Along the way, we shall retrieve the results of loc. cit., using our basis $\{p_\pi\}$. In particular, we shall give a geometric characterization of "admissible pairs" of loc. cit. (cf. Definition 16 below).

**Definition 12.** *Let $\lambda$ be a dominant weight. We say, $\lambda$ is of classical type if $\langle \lambda, \beta^\vee \rangle \leq 2$, for all $\beta \in R^+$.*

**Remarks.**

(1) Any dominant weight of classical type is either fundamental, or a sum of two minuscule fundamental weights.
(2) $G$ is classical if and only if all fundamental weights of $G$ are of classical type.

For the rest of this section, we fix a dominant weight $\lambda$ of classical type.

**Proposition 8.** *Let $v, w \in W^\lambda, v \leq w$. Then the $T$–module $H_w^v(\lambda)$ is at most one–dimensional; further, if non–zero, then it has the weight $-\frac{1}{2}(w(\lambda) + v(\lambda))$.*

*As a consequence, the weights of the $T$–module $H^0(X_w, L_\lambda(-\partial X_w))$ are among the $-\frac{1}{2}(w(\lambda) + x(\lambda))$ where $x \leq w$, and the corresponding weight spaces are one–dimensional.*

*Proof.* Let $p \in H_w^v(\lambda)$. Then $p^2$ belongs to $H^0(X_w^v, L_{2\lambda})$, and vanishes of order $\geq 2$ along each component of the whole boundary $(\partial X_w)^v \cup (\partial X^v)_w$. On the other hand, the product $p_w p_v$ also belongs to $H^0(X_w^v, L_{2\lambda})$ and satisfies by Chevalley's formula:

$$\operatorname{div}(p_w p_v) = \sum_\beta \langle \lambda, \beta^\vee \rangle X_{ws_\beta}^v + \sum_\gamma \langle \lambda, \gamma^\vee \rangle X_w^{vs_\gamma},$$

where $X_{ws_\beta}$ (resp. $X^{vs_\gamma}$) runs over all the components $X_x$ (resp. $X^y$) of $\partial X_w$ (resp. $\partial X^v$) such that $x \geq v$ (resp. $y \leq w$). Hence, $p_w p_v$ vanishes of order at most 2 along each component of $(\partial X_w)^v \cup (\partial X^v)_w$ (since $\lambda$ is of classical type), and nowhere else. Thus, $\frac{p^2}{p_w p_v}$ (a rational function on $X_w^v$) has no poles. It follows that $p^2 = c p_w p_v, c \in k$, and hence that $p$ is unique up to scalars; further, $p$ is either zero or has weight $\frac{1}{2}$(weight $p_w$+weight $p_v$) $= -\frac{1}{2}(w(\lambda) + v(\lambda))$. □

As a corollary to the proof of the above Proposition, we have

**Lemma 11.** *Let $v, w \in W^\lambda, v \leq w$. Further, let $H_w^v(\lambda)$ be non–zero. Then for each divisor $X_{ws_\beta}$ (resp. $X^{vs_\gamma}$) of $X_w$ (resp. $X^v$) such that $ws_\beta \geq v$ (resp. $vs_\gamma \leq w$), $\beta$ (resp. $\gamma$) being in $R^+$, we have, $\langle \lambda, \beta^\vee \rangle$ (resp. $\langle \lambda, \gamma^\vee \rangle$) $= 2$.*

We shall denote by $p_{w,v}$ the unique $p_{w,v}^\xi$, if non–zero (then $p_{w,w} = p_w$). By Proposition 8, $p_{w,v}$ lifts to a unique $T$–eigenvector in $H^0(X_w, L_\lambda(-\partial X_w))$; we still denote that lift by $p_{w,v}$. The non–zero $p_{w,v}$, where $v \leq w$, form a basis of $H^0(X_w, L_\lambda(-\partial X_w))$.

Notice that $\frac{p_{w,v}^2}{p_w}$ is a rational section of $L_\lambda$ on $X_w$, eigenvector of $T$ with weight $-v(\lambda)$, and without poles by the argument of Proposition 8. This implies

**Lemma 12.** *With notations as above, we have $p_{w,v}^2 = p_w p_v$ on $X_w$, up to a non–zero scalar.*

We now aim at characterizing those pairs $(v, w)$ such that $p_{w,v} \neq 0$. For this, we recall some definitions and Lemmas from [9].

**Definition 13.** *Let $X_v$ be a Schubert divisor in $X_w$; further, let $v = s_\alpha w$ where $\alpha \in R^+$. If $\alpha$ is simple, then we say, $X_v$ is a moving divisor in $X_w$, moved by $\alpha$.*

**Lemma 13.** *(*[9] *Lemma 1.5.) Let $X_v$ be a moving divisor in $X_w$, moved by $\alpha$. Let $X_u$ be any Schubert subvariety of $X_w$. Then either, $X_u \subseteq X_v$ or $X_{s_\alpha u} \subseteq X_v$.*

**Definition 14.** *Let $v, w \in W^\lambda, v \leq w, \ell(v) = \ell(w) - 1$; further let $v = ws_\beta = s_\gamma w$, for some positive roots $\beta, \gamma$. We denote the positive integer $\langle \lambda, \beta^\vee \rangle (= \langle v(\lambda), \gamma^\vee \rangle = -\langle w(\lambda), \gamma^\vee \rangle)$ by $m_\lambda(v, w)$, and refer to it as the Chevalley multiplicity of $X_v$ in $X_w$ (see* [3]*).*

**Lemma 14.** *(*[9] *Lemma 2.5.) Let $v, w \in W^\lambda$ such that $X_v$ is a moving divisor in $X_w$, moved by $\alpha$. Let $X_u$ be another Schubert divisor in $X_w$. Then $X_{s_\alpha u}$ is a divisor in $X_v$, and $m_\lambda(s_\alpha u, v) = m_\lambda(u, w)$.*

**Definition 15.** *Let $v, w \in W^\lambda$ such that $X_v$ is a divisor in $X_w$. If $m_\lambda(v, w) = 2$, then we shall refer to $X_v$ as a double divisor in $X_w$.*

By Lemma 11, if $p_{w,v} \neq 0$, then all Schubert divisors in $X_w$ that meet $X^v$ are double divisors.

**Lemma 15.** *([9] Lemma 2.6.) Let $u, w \in W^\lambda$ such that $X_u$ is a double divisor in $X_w$. Then $X_u$ is a moving divisor in $X_w$.*

**A geometric characterization of Admissible pairs:** Recall (cf.[9]):

**Definition 16.** *A pair $(v, w)$ in $W^\lambda$ is called admissible if either $v = w$ (in which case, it is called a trivial admissible pair), or there exists a sequence $w = w_1 > w_2 > \cdots > w_r = v$, such that $X_{w_{i+1}}$ is a double divisor in $X_{w_i}$, i.e., $m_\lambda(w_{i+1}, w_i) = 2$. We shall refer to such a chain as a double chain.*

We shall give a geometric characterization of admissible pairs (cf. Proposition 9 below). First we prove some preparatory Lemmas.

**Lemma 16.** *Let $p_{w,v} \neq 0$, then*

(1) *For any double divisor $X_{s_\alpha w}$ in $X_w$ meeting $X^v$, we have*
$$p_{s_\alpha w,v} = e_{-\alpha} p_{w,v} \text{ and } e_{-\alpha}^2 p_{w,v} = 0,$$
*where $e_{-\alpha}$ is a generator of the Lie algebra of $U_{-\alpha}$. Further, $p_{s_\alpha w,v} \neq 0$.*

(2) *Likewise, for any double divisor $X^{s_\alpha v}$ in $X^v$ meeting $X_w$, we have*
$$p_{w,s_\alpha v} = e_\alpha p_{w,v} \text{ and } e_\alpha^2 p_{w,v} = 0,$$
*where $e_\alpha$ is a generator of the Lie algebra of $U_\alpha$. Further, $p_{w,s_\alpha v} \neq 0$.*

(3) *The pair $(v, w)$ is admissible.*

*Proof.* (1) Consider the $T$–module $H^0(X_w^v, L_\lambda(-(\partial X^v)_w))$. By Proposition 8, it has a basis $\{p_{x,v} | \, v \leq x \leq w\}$ with corresponding weights $-\frac{1}{2}(x(\lambda) + v(\lambda))$. Notice that $X_w$ is invariant under $U_{-\alpha}$ (since $s_\alpha w < w$); hence $X_w^v$ and $(\partial X^v)_w$ are also $U_{-\alpha}$–invariant. Thus, $U_{-\alpha}$ acts on $H^0(X_w^v, L_\lambda(-(\partial X^v)_w))$, compatibly with the $T$–action. The $U_{-\alpha}$–submodule $M$ generated by $p_{w,v}$ is $T$–invariant, with weights of the form $-\frac{1}{2}(w(\lambda) + v(\lambda)) - m\alpha$ for some non–negative integers $m$. But if $x(\lambda) = w(\lambda) + 2m\alpha$, then either $x = w$ and $m = 0$, or $x = s_\alpha w$ and $m = 1$ (by Lemma 11). Hence $M$ is either spanned by $p_{w,v}$, or by $p_{w,v}$ and $p_{s_\alpha w,v}$. Further, $e_{-\alpha}^2 p_{w,v} = 0$.

To complete the proof, it suffices to show that $U_{-\alpha}$ does not fix $p_{w,v}$. Otherwise, the zero locus of $p_{w,v}$ in $X_w^v$ is $U_{-\alpha}$–invariant, and hence so is $(\partial X_w)^v$. Thus,
$$\overline{U_{-\alpha} e_{s_\alpha w}} \subseteq (\partial X_w)^v.$$
But $e_w \in \overline{U_{-\alpha} e_{s_\alpha w}}$ (since $s_\alpha w < w$) and $e_w \notin (\partial X_w)^v$, a contradiction.

(2) is checked similarly. And (3) follows from (1) together with Lemma 11, by induction on $\ell(w)$. ☐

**Lemma 17.** *Let $(v, w)$ be an admissible pair, then $p_{w,v} \neq 0$.*

*Proof.* We argue by induction on $\ell(w)$. We may chose a simple root $\alpha$ such that $w > s_\alpha w \geq v$ and that $X_{ws_\alpha}$ is a double divisor in $X_w$. Then $\langle w(\lambda), \check{\alpha}\rangle = -2$, and also $p_{s_\alpha w, v} \neq 0$ by the induction hypothesis. The weight of this vector is

$$-\frac{1}{2}(s_\alpha w(\lambda) + v(\lambda)) = -\frac{1}{2}(w(\lambda) + v(\lambda)) - \alpha.$$

The scalar product of this weight with $\check{\alpha}$ being integral, $\langle v(\lambda), \check{\alpha}\rangle$ is an even integer. Since $\lambda$ is of classical type, it follows that

$$\langle v(\lambda), \check{\alpha}\rangle \in \{2, 0, -2\}.$$

We now distinguish the following three cases:

**Case 1:** $(v(\lambda), \alpha^\vee) = 2$. Then $w \geq s_\alpha w, s_\alpha v > v$. As a first step, we find a relation between $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))$ and $H^0(X_{s_\alpha w}^v, L_\lambda(-(\partial X_{s_\alpha w})^v))$.

Let $G_\alpha$ be the subgroup of $G$ generated by $U_\alpha$, $U_{-\alpha}$ and $T$; let $B_\alpha = G_\alpha \cap B$. Then the derived subgroup of $G_\alpha$ is isomorphic to $SL(2)$ or to $PSL(2)$, and $G_\alpha/B_\alpha$ is isomorphic to the projective line $\mathbb{P}^1$. For a $B_\alpha$–module $M$, we shall denote the associated $G_\alpha$–linearized locally free sheaf on $G_\alpha/B_\alpha$ by $\underline{M}$.

Notice that $X_w$, $X^v$ and hence $X_w^v$ are invariant under $G_\alpha$, and $(\partial X_w)^v$ is invariant under $B_\alpha$; we have

$$(\partial X_w)^v = X_{s_\alpha w}^v \cup G_\alpha(\partial X_{s_\alpha w})^v.$$

Consider the fiber product $G_\alpha \times^{B_\alpha} X_{s_\alpha w}^v$ with projection

$$p : G_\alpha \times^{B_\alpha} X_{s_\alpha w}^v \longrightarrow G_\alpha/B_\alpha \simeq \mathbb{P}^1$$

and "multiplication" map

$$\psi : G_\alpha \times^{B_\alpha} X_{s_\alpha w}^v \longrightarrow X_w^v.$$

Then $\psi$ is birational (since it is an isomorphism at $e_w$). Further, we have

$$(\partial X_w)^v = \psi(X_{s_\alpha w}^v \cup G_\alpha \times^{B_\alpha} (\partial X_{s_\alpha w})^v)$$

where $X_{s_\alpha w}^v$ is the fiber of $p$ at $B_\alpha/B_\alpha$. By the projection formula, it follows that

$$L_\lambda(-(\partial X_w)^v) = \psi_*\psi^* L_\lambda(-X_{s_\alpha w}^v - G_\alpha \times^{B_\alpha} (\partial X_{s_\alpha w})^v).$$

This yields an isomorphism

$$H^0(X_w^v, L_\lambda(-(\partial X_w)^v)) \cong H^0(G_\alpha/B_\alpha, p_*\psi^* L_\lambda(-X_{s_\alpha w}^v - G_\alpha \times^{B_\alpha} (\partial X_{s_\alpha w})^v)).$$

Further, we may identify the $G_\alpha$–linearized sheaf $p_*\psi^* L_\lambda(-G_\alpha \times^{B_\alpha} (\partial X_{s_\alpha w})^v))$ on $G_\alpha/B_\alpha$, to the sheaf $\underline{H^0(X_{s_\alpha w}^v, L_\lambda(-(\partial X_{s_\alpha w})^v))}$. Therefore, we obtain an exact sequence of $B_\alpha$–modules

$$0 \longrightarrow H^0(X_w^v, L_\lambda(-(\partial X_w)^v)) \longrightarrow H^0(G_\alpha/B_\alpha, \underline{H^0(X_{s_\alpha w}^v, L_\lambda(-(\partial X_{s_\alpha w})^v))}) \longrightarrow H^0(X_{s_\alpha w}^v, L_\lambda(-(\partial X_{s_\alpha w})^v)) \longrightarrow 0,$$

where the map on the right is the "evaluation" map (its surjectivity follows e.g. from Corollary 1.)

Next we analyse the $B_\alpha$–module $H^0(X^v_{s_\alpha w}, L_\lambda(-(\partial X_{s_\alpha w})^v))$. By Proposition 8, its weights have multiplicity one; they are among the $-\frac{1}{2}(s_\alpha w(\lambda) + x(\lambda))$, where $v \le x \le s_\alpha w$, and the weight $-\frac{1}{2}(w(\lambda) + v(\lambda)) - \alpha$ occurs, since $p_{s_\alpha w,v} \neq 0$; its $\alpha$–weight (the scalar product with $\check{\alpha}$) is $-2$.

If $s_\alpha v \not\le s_\alpha w$, then the span $M$ of $p_{s_\alpha w,v}$ is invariant under $B_\alpha$. Thus, the $T$–module $H^0(G_\alpha/B_\alpha, \underline{M})$ has weights $-\frac{1}{2}(w(\lambda) + v(\lambda)) + (m-1)\alpha$, $m = 0, 1, 2$, each of them having multiplicitly one. Further, the kernel of the evaluation map $H^0(G_\alpha/B_\alpha, \underline{M}) \longrightarrow M$ contains an element of weight $-\frac{1}{2}(w(\lambda)+v(\lambda))$. By the exact sequence above, this weight occurs in $H^0(X^v_w, L_\lambda(-(\partial X_w)^v))$; using Proposition 8 again, it follows that $p_{w,v} \neq 0$.

On the other hand, if $s_\alpha v \le s_\alpha w$, then Lemma 16 (2) applied to $(s_\alpha w, v)$ yields

$$p_{s_\alpha w, s_\alpha v} = e_\alpha p_{s_\alpha w, v} \neq 0.$$

Hence the span $M$ of $p_{s_\alpha w,v}$ and $p_{s_\alpha w,s_\alpha v}$ is a non–trivial $B_\alpha$-module with $\alpha$-weights $-2$ and $0$ (note that $e_\alpha p_{s_\alpha w,s_\alpha v} = 0$ by weight considerations). Thus, we have an isomorphism of $B_\alpha$–modules

$$M \cong M_1 \otimes M_2,$$

where $M_1$ is a one–dimensional $B_\alpha$–module with $\alpha$–weight $-1$, and $M_2$ is the standard two–dimensional $G_\alpha$–module. It follows that the weights of the $T$–module

$$H^0(G_\alpha/B_\alpha, \underline{M}) \cong H^0(G_\alpha/B_\alpha, \underline{M_1}) \otimes M_2$$

are exactly $-\frac{1}{2}(w(\lambda) + v(\lambda)) - \alpha$, $-\frac{1}{2}(w(\lambda) + v(\lambda)) + \alpha$ (both of multiplicity one) and $-\frac{1}{2}(w(\lambda) + v(\lambda))$ (of multiplicity two). Thus, the kernel of the evaluation map $H^0(G_\alpha/B_\alpha, \underline{M}) \longrightarrow M$ contains an element of weight $-\frac{1}{2}(w(\lambda) + v(\lambda))$, and we conclude as above.

**Case 2:** $\langle v(\lambda), \check{\alpha}\rangle = 0$. Then $w > s_\alpha w \ge v = s_\alpha v$, so that $X_w$, $X^v$ and $X^v_w$ are again invariant under $G_\alpha$, whereas $(\partial X_w)^v$ is $B_\alpha$–invariant. Arguing as in Case 1, we obtain the same relation between $H^0(X^v_w, L_\lambda(-(\partial X_w)^v))$ and $H^0(X^v_{s_\alpha w}, L_\lambda(-(\partial X_{s_\alpha w})^v))$; but now the latter $B_\alpha$–module contains the span $M$ of $p_{s_\alpha w,v}$, as a $B_\alpha$-submodule of $\alpha$–weight $-1$. As in Case 1, it follows that $p_{w,v} \neq 0$.

**Case 3:** $\langle v(\lambda), \check{\alpha}\rangle = -2$. Then $w > s_\alpha w \ge v > s_\alpha v$, and $X^v$ is a double divisor in $X^{s_\alpha v}$. Therefore, the pair $(s_\alpha v, s_\alpha w)$ is admissible. By the induction hypothesis, we have, $p_{s_\alpha v, s_\alpha w} \neq 0$. Then Case 1 applies to the pair $(s_\alpha v, w)$ and yields $p_{w,s_\alpha v} \neq 0$. Further, $X^v$ is a double divisor in $X^{s_\alpha v}$. Hence by Lemma 16 (2) applied to $(w, s_\alpha v)$, we obtain $p_{w,v} \neq 0$. $\square$

Now combining Lemmas 11, 16 and 17, we obtain

**Proposition 9.** *Let $v, w \in W^\lambda, v \le w$. Then the pair $(v,w)$ is admissible if and only if $p_{w,v}$ is non–zero. In this case, every chain from $v$ to $w$ is a double chain.*

## 7. Standard monomials for sums of weights of classical type

In this section, we obtain a standard monomial basis for $H^0(X_w^v, L_{\lambda_1+\cdots+\lambda_m})$, where $X_w^v$ is a Richardson variety in $G/P$, and $\lambda_1, \ldots, \lambda_m$ are dominant characters of classical type of $P$ (in the sense of Definition 12).

We begin with the case where $m = 1$; we shall need a definition, and a result of Deodhar ([9] Lemmas 4.4 and 4.4') on the Bruhat ordering.

**Definition 17.** *Let $w \in W^P$ and let $\lambda$ be a dominant character of $P$. We say that $x \in W^P$ is $\lambda$–maximal in $w$ (resp. $\lambda$–minimal on $w$) if $xy \le x$ for any $y \in W_\lambda$ such that $xy \in W^P$ and $xy \le w$ (resp. if $xy \ge x$ for any $y \in W_\lambda$ such that $xy \in W^P$ and $xy \ge w$).*

**Lemma 18.** *Let $w \in W$ and $x \in W^\lambda$ such that $x \le w(\lambda)$ (resp. $x \ge v(\lambda)$). Then the set $\{y \in W_\lambda \mid xy \le w\}$ (resp. $\{y \in W_\lambda \mid w \le xy\}$ admits a unique maximal (resp. minimal) element.*

We shall also need the following consequences of this result.

**Lemma 19.** (1) *Let $w \in W^P$ and $x \in W^\lambda$ such that $x \le w(\lambda)$ (resp. $x \ge w(\lambda)$). Then $x \in W/W_\lambda$ admits a unique lift $\tilde{x} \in W^P$ such that $\tilde{x}$ is $\lambda$–maximal in $w$ (resp. $\lambda$–minimal on $w$).*
(2) *Let $v \le w \in W^P$, then $v$ is $\lambda$–maximal in $w$ (resp. $w$ is $\lambda$–minimal in $v$) if and only if $(\partial_\lambda X^v)_w = (\partial X^v)_w$ (resp. $(\partial_\lambda X_w)^v = (\partial X_w)^v$).*

*Proof.* (1) Let $x \le w(\lambda)$. By Lemma 18, the set $\{y \in W_\lambda \mid xy \le w\}$ admits a unique maximal element that we still denote by $y$. Let $\tilde{x}$ be the representative in $W^P$ of $xy \in W$, then $\tilde{x}(\lambda) = xy(\lambda) = x(\lambda)$. Further, if we have $\tilde{x}z \le w$ for some $z \in W_\lambda$ such that $\tilde{x}z \in W^P$, then we can write $\tilde{x}z = xu$ where $u \in W_\lambda$. Since $xu \le w$, we have $u \le y$ and hence $xu \le xy$ (since $x \in W^\lambda$ and $u, y \in W_\lambda$). But $xu = \tilde{x}z \in W^P$, so that $\tilde{x}z \le \tilde{x}$. This proves the assertion concerning $\lambda$–maximal elements, and hence the dual assertion concerning $\lambda$–minimal elements.

(2) If $(\partial_\lambda X^v)_w \ne (\partial X^v)_w$, then there exists $y \in W_\lambda$ such that $v < vy \le w$ and $\ell(vy) = \ell(v) + 1$. Thus, $v$ is not $\lambda$–maximal in $w$.

Conversely, if $v$ is not $\lambda$–maximal in $w$, then $v < \tilde{v} \le w$ where $\tilde{v} \in vW_\lambda$ is $\lambda$–maximal in $w$. Hence there exists $y \in W_\lambda$ such that $v < vy \le \tilde{v} \le w$ and $\ell(vy) = \ell(v) + 1$. Now $X_w^{vy}$ is contained in $(\partial X^v)_w$ but not in $(\partial_\lambda X^v)_w$. □

Now we consider the $T$–module $H^0(X_w^v, L_\lambda)$, where $v \leq w \in W^P$ and $\lambda$ is a dominant character of $P$, not necessarily $P$–regular. Notice that the diagram

$$\begin{array}{ccc} H^0(X_{w(\lambda)}, L_\lambda) & \longrightarrow & H^0(X_{w(\lambda)}^{v(\lambda)}, L_\lambda) \\ \downarrow & & \downarrow \\ H^0(X_w, L_\lambda) & \longrightarrow & H^0(X_w^v, L_\lambda) \end{array}$$

is commutative, where the horizontal (resp. vertical) maps are restrictions (resp. pull–backs). Further, both restrictions are surjective by Proposition 1; and the pull–back on the left is an isomorphism, since the natural map $f : X_w \to X_{w(\lambda)}$ satisfies $f_*\mathcal{O}_{X_w} = \mathcal{O}_{X_{w(\lambda)}}$. Thus, we may regard the $T$–module $H^0(X_w^v, L_\lambda)$ as a quotient of $H^0(X_{w(\lambda)}^{v(\lambda)}, L_\lambda)$.

Likewise, by using the commutative diagram

$$\begin{array}{ccc} H^0(X_{w(\lambda)}, L_\lambda(-\partial X_{w(\lambda)})) & \longrightarrow & H^0(X_{w(\lambda)}^{v(\lambda)}, L_\lambda(-(\partial X_{w(\lambda)})^{v(\lambda)})) \\ \downarrow & & \downarrow \\ H^0(X_w, L_\lambda(-\partial_\lambda X_w)) & \longrightarrow & H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v)) \end{array}$$

and Corollary 1, we may regard the $T$–module $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$ as a quotient of $H^0(X_{w(\lambda)}^{v(\lambda)}, L_\lambda(-(\partial X_{w(\lambda)})^{v(\lambda)}))$. The latter has been described in Section 6, in the case that $\lambda$ is of classical type: it has a basis consisting of the $p_{w(\lambda),x(\lambda)}$ where $x(\lambda) \in W^\lambda$, $v(\lambda) \leq x(\lambda) \leq w(\lambda)$ and the pair $(x(\lambda), w(\lambda))$ is admissible. From that description we shall deduce

**Proposition 10.** *Let $v \leq w \in W^P$ and let $\lambda$ be a dominant character of classical type of $P$.*

(1) *The space $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v - (\partial X^v)_w))$ is spanned by $p_{w(\lambda),v(\lambda)}$, if $v$ is $\lambda$–maximal in $w$; otherwise, this space is zero.*
(2) *The $p_{w(\lambda),x(\lambda)}$ where $x \in W^P$ and $v \leq x \leq w$, form a basis of the space $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$.*
(3) *The $p_{w(\lambda),x(\lambda)}$ where $x \in W^P$ is $\lambda$–minimal on $v$, and $w$ is $\lambda$–minimal on $x$, form a basis of $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))$.*
(4) *The $p_{y(\lambda),x(\lambda)}$ where $x, y \in W^P$ and $v \leq x \leq y \leq w$, form a basis of $H^0(X_w^v, L_\lambda)$.*

*Proof.* (1) Assume that $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v - (\partial X^v)_w))$ contains a non–zero element $p$. Then, by the argument of Proposition 8, $\frac{p^2}{p_{w(\lambda)}p_{v(\lambda)}}$ is a rational function on $X_w^v$, without poles; further, it vanishes identically on $(\partial X^v)_w - (\partial_\lambda X^v)_w$, since the zero locus of $p_{v(\lambda)}$ is $(\partial_\lambda X^v)_w$. It follows that $p^2$ is a constant multiple of $p_{w(\lambda)}p_{v(\lambda)}$, and that $(\partial X^v)_w = (\partial_\lambda X^v)_w$. Hence $p$ is a constant multiple of $p_{w(\lambda),v(\lambda)}$, and $v$ is $\lambda$–maximal in $w$ (by Lemma 19).

Conversely, let $v$ be $\lambda$–maximal in $w$; then $(\partial_\lambda X^v)_w = (\partial X^v)_w$. Thus, $p_{w(\lambda),v(\lambda)}$ vanishes identically on $(\partial_\lambda X_w)^v \cup (\partial X^v)_w$. Further, $p_{w(\lambda),v(\lambda)} \neq 0$ on $X_w^v$, since $p^2_{w(\lambda),v(\lambda)} = p_{w(\lambda)}p_{v(\lambda)}$ on $X_w$ (by Lemma 12).

(2) By Proposition 8, the space $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$ is spanned by the images of the $p_{w(\lambda),x(\lambda)}$ where $v(\lambda) \leq x(\lambda) \leq w(\lambda)$. Further, $p^2_{w(\lambda),x(\lambda)} = p_{w(\lambda)}p_{x(\lambda)}$ on $X_w$. Using Lemma 2, we see that $p_{w(\lambda),x(\lambda)}$ is non–zero on $X_w^v$ if and only if $x(\lambda)$ has a representative $x \in W^P$ such that $v \leq x \leq w$.

(3) $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))$ is a $T$–stable subspace of $H^0(X_w^v, L_\lambda(-(\partial_\lambda X_w)^v))$; thus, it is spanned by certain $p_{w(\lambda),x(\lambda)}$ where $v \leq x \leq w$. By Lemma 12, the zero locus $(p_{w(\lambda),x(\lambda)} = 0)$ in $X_w^v$ equals $(p_{x(\lambda)} = 0) \cup (\partial_\lambda X_w)^v$. Hence $p_{w(\lambda),x(\lambda)}$ belongs to $H^0(X_w^v, L_\lambda(-(\partial X_w)^v))$ if and only if $p_{x(\lambda)}$ vanishes identically on $(\partial X_w)^v - (\partial_\lambda X_w)^v$. By Lemma 2, this amounts to: $x(\lambda)$ admits no lift $x'$ such that $v \leq x' \leq wy$ for some $y \in W_\lambda$, $wy < y$, $\ell(wy) = \ell(w) - 1$. Let $\tilde{x}$ be the lift of $x(\lambda)$ that is $\lambda$–minimal on $v$, then the preceding condition means that $w$ is $\lambda$–minimal on $\tilde{x}$.

(4) By Proposition 3, we obtain a basis of the space $H^0(X_w^v, L_\lambda)$ by choosing a basis of $H^0(X_w^x, L_\lambda(-(\partial X^x)_w))$ for each $x \in W^P$ such that $v \leq x \leq w$, and lifting this basis to $H^0(X_w^v, L_\lambda)$ under the (surjective) restriction map $H^0(X_w^v, L_\lambda) \to H^0(X_w^x, L_\lambda)$. Together with (3), it follows that a basis of $H^0(X_w^v, L_\lambda)$ consists of the $p_{y(\lambda),x(\lambda)}$, where $y$ is $\lambda$–maximal in $w$, $x$ is $\lambda$–maximal in $y$, and $v \leq x$. But given any $x', y' \in W^P$ such that $v \leq x' \leq y' \leq w$, we have $v \leq x \leq y \leq w$ and $p_{y'(\lambda),x'(\lambda)} = p_{y(\lambda),x(\lambda)}$, where $x$ (resp. $y$) is the representative of $x(\lambda)$ that is $\lambda$–maximal in $w$ (resp. $x$). □

**Definition 18.** *Let $\lambda_1, \ldots, \lambda_m$ be dominant characters of classical type of $P$. Let $\pi_i = (w_i, v_i)$ where $v_i \leq w_i \in W^{\lambda_i}$ for $1 \leq i \leq m$. Then the sequence $\underline{\pi} = (\pi_1, \ldots, \pi_m)$ is standard if there exist lifts $\tilde{w}_i, \tilde{v}_i$ in $W^P$ for $1 \leq i \leq m$, such that*

$$\tilde{v}_m \leq \tilde{w}_m \leq \cdots \leq \tilde{v}_1 \leq \tilde{w}_1.$$

*The monomial*

$$p_{\underline{\pi}} = p_{w_1(\lambda_1),v_1(\lambda_1)} \cdots p_{w_m(\lambda_m),v_m(\lambda_m)} \in H^0(G/P, L_{\lambda_1+\cdots+\lambda_m})$$

*is called standard as well.*

*Further, let $v, w \in W^P$ such that $v \leq w$; then $\underline{\pi}$ is standard on $X_w^v$ if there exist lifts as above, such that*

$$v \leq \tilde{v}_m \leq \tilde{w}_m \leq \cdots \leq \tilde{v}_1 \leq \tilde{w}_1 \leq w.$$

*The restriction of $p_{\underline{\pi}}$ to $X_w^v$ is called a standard monomial on $X_w^v$; it is a $T$–eigenvector in $H^0(X_w^v, L_{\lambda_1+\cdots+\lambda_m})$.*

Notice that there is no loss of generality in assuming that $\tilde{v}_m$ is $\lambda_m$–minimal on $v$, and that $\tilde{w}_m$ is $\lambda_m$–minimal on $\tilde{v}_m$.

Now the argument of Proposition 7, together with Proposition 10 and induction on $m$, yields the following partial generalization of Corollary 3.

**Theorem 3.** *Let $v \leq w \in W^P$ and let $\lambda_1, \ldots, \lambda_m$ be dominant characters of $P$. If $\lambda_1, \ldots, \lambda_m$ are of classical type, then the standard monomials on $X_w^v$ form a basis for $H^0(X_w^v, L_{\lambda_1 + \cdots + \lambda_m})$.*

**Remarks.**

(1) In particular, Theorem 3 applies to $P = B$ if all fundamental weights are of classical type, that is, if $G$ is classical. Thereby, we retrieve all results of [9].

(2) The second assertion of Corollary 3 does not generalize to this setting, that is, there are examples of standard monomials on $G/P$ which are not standard on $X_w^v$, but which restrict non–trivially to that subvariety.

Specifically, let $G = SL(3)$ with simple reflections $s_1, s_2$ and fundamental weights $\omega_1, \omega_2$. Then one may check that the monomial

$$p_{s_1(\omega_1)} p_{s_2(\omega_2)} \in H^0(G/B, L_{\omega_1+\omega_2})$$

is standard on $G/B$ and restricts non–trivially to $X_{s_2 s_1}$, but is not standard there.

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