

SCHUBERT VARIETIES, TORIC VARIETIES, AND LADDER DETERMINANTAL VARIETIES

N. GONCIULEA AND V. LAKSHMIBAI[†]

ABSTRACT. We construct certain normal toric varieties (associated to finite distributive lattices) which are degenerations of the Grassmannians. We also determine the singular loci for certain normal toric varieties, namely the ones which are certain ladder determinantal varieties. As a consequence, we prove the conjecture of [?] on the components of the singular locus, for certain Schubert varieties in the flag variety.

INTRODUCTION

The results of this paper are two-folds. On the one hand, we give a self-contained exposition of the results of [?] on the degenerations of the Grassmannians to normal toric varieties. On the other hand, we determine the irreducible components of the singular locus of the determinantal variety (in an one-sided ladder) defined by the vanishing of all 2 minors. Using the latter result, we prove the conjecture in [?] on the irreducible components of the singular locus for certain Schubert varieties in the flag variety $SL(n)/B$. Let $G_{d,n}$ be the Grassmannian of d -planes in k^n , k being the base field, which is assumed to be algebraically closed of arbitrary characteristic. For $G_{d,n}$, as well as its Schubert varieties, we proved in [?] their degenerations to normal toric varieties. The normality of these toric varieties was concluded in [?] using the results of [?]. In this paper, among other things, we give an independent proof of normality of these toric varieties, thus leading to a self-contained exposition of the results of [?] for $G_{d,n}$. This is accomplished by showing that these varieties are Cohen-Macaulay, and are nonsingular in codimension 1 (using the Jacobian Criterion for smoothness, cf. Sections ?? and ??). We also prove some partial results (cf. Section ??) towards the determination of the irreducible components of the singular loci of these varieties, and state a conjecture on the components of the singular loci. For the case $d = 2$, these toric varieties are in fact the determinantal varieties (in some one-sided

[†]Partially supported by NSF Grant DMS 9502942

[†]Partially supported by Northeastern University RSDF 95-96.

For the case $\mathcal{L} = \mathcal{I}_{[\cdot, \setminus]}$, where $I_{d,n} = \{\underline{i} = (i_1, \dots, i_d) \mid 1 \leq i_1 < \dots < i_d \leq n\}$, $R(\mathcal{L})$ is realized in [?] as the special fiber of a certain flat family whose general fiber is $k[G_{d,n}]$, the homogeneous coordinate ring of the Grassmannian $G_{d,n}$ for the Plücker embedding. We prove (cf. Section ??) that the variety $X(\mathcal{L}) = \text{Spec} \mathcal{R}(\mathcal{L})$ is nonsingular in codimension 1. (This is accomplished by using the Jacobian criterion for smoothness). Thus we obtain the normality of $X_{d,n}$. As a consequence, we obtain (see Theorem ??):

Theorem A . *The Grassmannian $G_{d,n}$ degenerates to the normal toric variety $X_{d,n}$.*

Regarding the other half of the results of this paper, let L be an one-sided ladder, and $X(L)$ the determinantal variety defined by the vanishing of all the 2 minors in L . Let $\alpha_1, \dots, \alpha_l$ be all the upper outer corners of L , and α_0 the unique lower outer corner of \mathcal{L} (see Figure 1). For $1 \leq i \leq l$, let V_i be the subvariety of $X(L)$ defined by $x_\alpha = 0$ for $\alpha \in [\alpha_0 \wedge \alpha_i, \alpha_0 \vee \alpha_i]$ (see Section ?? for notations).

In Section ?? we prove the following result (see Theorem ??):

Theorem B . *The irreducible components of the singular locus of $X(L)$ are precisely V_i , $1 \leq i \leq l$.*

This is again proved using the Jacobian Criterion.

As mentioned above, $X(L) \times \mathbb{A}^\geq$ gets identified with the “opposite cell” in $X(w)$, for a certain Schubert variety $X(w)$ in G/Q , $G = SL(n)$, for a suitable n and a certain parabolic subgroup Q . Moreover, the varieties $V_i \times \mathbb{A}^\geq$, $1 \leq i \leq l$, also get identified with certain Schubert subvarieties $X(\theta_i)$ of $X(w)$ (see Section ?? for details). Let $X(w^{\max}) = \pi^{-1}(X(w))$, where π is the canonical projection $G/B \rightarrow G/Q$. In Section ?? we prove the following result (see Theorem ??):

Theorem C . *The conjecture in [?] (on the irreducible components of the singular locus) holds for $X(w^{\max})$.*

(see Section ?? for the statement of the conjecture).

The sections are organized as follows. In Sections ?? and ??, we recall some generalities on Gröbner bases and distributive lattices. In Section ??, we prove the primality of the ideal $I(\mathcal{L})$, \mathcal{L} being any finite distributive lattice (our proof is very short, and combinatorial in nature). In Section ??, we carry out a short geometric proof of the fact that a binomial prime ideal is toric. In Section ??, we prove some general results on $X(\mathcal{L})$ (in particular, we compute the dimension of $X(\mathcal{L})$). In Section ??, we construct a “standard monomial basis” for $R(\mathcal{L})$. In Section ??, we prove the Cohen-Macaulayness of $R(\mathcal{L})$. In Section ??, we derive some properties of the distributive lattice

$I_{d,n} = \{\underline{i} = (i_1, \dots, i_d) \mid 1 \leq i_1 < \dots < i_d \leq n\}$. In Section ??, we study the variety $X_{d,n}$, where $X_{d,n} = X(\mathcal{L})$, for $\mathcal{L} = \mathcal{I}_{[\cdot, \setminus]}$. In Section ??, we prove the normality of the variety $X_{d,n}$. In Section ??, we prove some partial results, and state a conjecture, on the irreducible components of $\text{Sing } X_{d,n}$. In Section ??, we recall the generalities on G/Q . In Section ??, we recall some results on the flag variety $SL(n)/B$, and its Schubert varieties. In Section ??, we verify the conjecture stated in Section ?? for $d = 2$, by determining the irreducible components of $\text{Sing } X(L)$, where $X(L)$ is the determinantal variety defined by the vanishing of all the 2 minors in a general one-sided ladder L . In Section ??, we relate $X(L)$, as well as the irreducible components of $\text{Sing } X(L)$ to opposite cells in certain Schubert varieties in certain $SL(n)/Q$. In Section ??, we verify the conjecture in [?] to be true for the pull-backs in G/B (under $G/B \rightarrow G/Q$) of the Schubert varieties obtained in Section ??.

It should be remarked that in [?] a purely combinatorial description of $\text{Sing } X(\mathcal{L})$ is obtained. In this paper we have taken a geometric approach to this problem, in the case when $\mathcal{L} = \mathcal{I}_{[\cdot, \setminus]}$, or \mathcal{L} is an one-sided ladder.

In this paper we have considered only those ladder determinantal varieties which are also toric varieties. In a subsequent paper [?], we study a larger class of ladder determinantal varieties (which are not necessarily toric varieties), and prove results similar to those of Sections ??, ?? and ??.

The authors wish to thank C. S. Seshadri and P. Sankaran for many useful discussions, especially pertaining Sections ?? and ??.

1. GENERALITIES ON GRÖBNER BASES

Let k be a field, and consider the ring $k[x_1, \dots, x_n]$ of polynomials in n variables x_1, \dots, x_n . We recall below some generalities on Gröbner bases; for more details one may refer to [?], [?].

Definition 1.1. A total order \preceq on the set of monomials in $k[x_1, \dots, x_n]$ is called a *monomial order* if for given monomials \underline{m} , \underline{m}_1 , \underline{m}_2 , with $\underline{m} \neq 1$, $\underline{m}_1 \prec \underline{m}_2$ implies $\underline{m}_1 \prec \underline{m} \cdot \underline{m}_1 \prec \underline{m} \cdot \underline{m}_2$.

For the rest of this Section, a fixed monomial order \preceq is considered.

If f is a nonzero polynomial in $k[x_1, \dots, x_n]$, then the greatest monomial (with respect to \preceq) occurring in f is called the *initial monomial of f* , and we denote it by $\text{in}(f)$; the coefficient of $\text{in}(f)$ is called the *initial coefficient of f* . For a family of polynomials $\mathcal{F} \subset \mathbb{K}[\mathfrak{s}_\infty, \dots, \mathfrak{s}_\setminus]$, the ideal generated by its elements will be denoted by $\langle \mathcal{F} \rangle$, and the

set of the initial monomials of all polynomials in \mathcal{F} will be denoted by $\text{in}(\mathcal{F})$.

Definition 1.2. Let $I \subset k[x_1, \dots, x_n]$ be an ideal. A finite set of polynomials $\mathcal{F} \subset \mathcal{I}$ is called a *Gröbner basis for I with respect to \preceq* if $\langle \text{in}(\mathcal{F}) \rangle = \langle \text{in}(\mathcal{I}) \rangle$.

Definition 1.3. A *reduced Gröbner basis for I with respect to \preceq* is a Gröbner basis \mathcal{F} for I with respect to \preceq such that the initial coefficients of the elements in \mathcal{F} are all 1, and for all $f \in \mathcal{F}$, none of the monomials present in f lies in $\langle \text{in}(\mathcal{F} \setminus \{f\}) \rangle$.

Proposition 1.4. Any Gröbner basis for I generates I as an ideal.

In the case when I is the defining ideal of an algebraic variety X , a Gröbner basis for I will be also called a *Gröbner basis for X* .

Proposition 1.5. A nonzero ideal $I \subset k[x_1, \dots, x_n]$ has a unique reduced Gröbner basis (with respect to a given monomial order).

1.6. Reverse lexicographic order. Assume that the variables x_1, \dots, x_n are totally ordered as follows: $x_1 < \dots < x_n$. A monomial \underline{m} of degree r in the polynomial ring $k[x_1, \dots, x_n]$ will be written in the form $\underline{m} = x_{i_1} \dots x_{i_r}$, with $1 \leq i_1 \leq \dots \leq i_r \leq n$. The *reverse lexicographic order* on the set of monomials $\underline{m} \in k[x_1, \dots, x_n]$ is denoted by \preceq_{rlex} , and defined as follows: $x_{i_1} \dots x_{i_r} \prec_{rlex} x_{j_1} \dots x_{j_s}$ if and only if either $r < s$, or $r = s$ and there exists an $l < r$ such that $i_1 = j_1, \dots, i_l = j_l, i_{l+1} < j_{l+1}$. It is easy to check that \preceq_{rlex} is a monomial order.

2. GENERALITIES ON DISTRIBUTIVE LATTICES

Definition 2.1. A *lattice* is a partially ordered set (\mathcal{L}, \leq) such that, for every pair of elements $x, y \in \mathcal{L}$, there exist elements $x \vee y$ and $x \wedge y$, called the *join*, respectively the *meet* of x and y , defined by:

$$\begin{aligned} x \vee y &\geq x, \quad x \vee y \geq y, \quad \text{and if } z \geq x \text{ and } z \geq y, \text{ then } z \geq x \vee y, \\ x \wedge y &\leq x, \quad x \wedge y \leq y, \quad \text{and if } z \leq x \text{ and } z \leq y, \text{ then } z \leq x \wedge y. \end{aligned}$$

It is easy to check that the operations \vee and \wedge are commutative and associative.

Definition 2.2. An element $z \in \mathcal{L}$ is called the *zero* of \mathcal{L} , denoted by $\hat{0}$, if $z \leq x$ for all x in \mathcal{L} . An element $z \in \mathcal{L}$ is called the *one* of \mathcal{L} , denoted by $\hat{1}$, if $z \geq x$ for all x in \mathcal{L} .

Definition 2.3. Given a lattice \mathcal{L} , a subset $\mathcal{L}' \subset \mathcal{L}$ is called a *sublattice* of \mathcal{L} if $x, y \in \mathcal{L}'$ implies $x \wedge y \in \mathcal{L}'$, $x \vee y \in \mathcal{L}'$.

Definition 2.4. Two lattices \mathcal{L}_∞ and \mathcal{L}_ϵ are *isomorphic* if there exists a bijection $\varphi : \mathcal{L}_\infty \rightarrow \mathcal{L}_\epsilon$ such that, for all $x, y \in \mathcal{L}_\infty$,

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y) \text{ and } \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y).$$

Definition 2.5. A lattice is called *distributive* if the following identities hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad (1)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \quad (2)$$

2.6. An example. Given an integer $n \geq 1$, $\mathcal{C}(\setminus)$ will denote the chain $\{1 < \dots < n\}$, and for $n_1, \dots, n_d > 1$, $\mathcal{C}(\setminus_\infty, \dots, \setminus_\Gamma)$ will denote the chain product lattice $\mathcal{C}(\setminus_\infty) \times \dots \times \mathcal{C}(\setminus_\Gamma)$ consisting of all d -tuples (i_1, \dots, i_d) , with $1 \leq i_1 \leq n_1, \dots, 1 \leq i_d \leq n_d$. For (i_1, \dots, i_d) , (j_1, \dots, j_d) in $\mathcal{C}(\setminus_\infty, \dots, \setminus_\Gamma)$, we define

$$(i_1, \dots, i_d) \leq (j_1, \dots, j_d) \iff i_1 \leq j_1, \dots, i_d \leq j_d.$$

We have

$$(i_1, \dots, i_d) \vee (j_1, \dots, j_d) = (\max\{i_1, j_1\}, \dots, \max\{i_d, j_d\})$$

$$(i_1, \dots, i_d) \wedge (j_1, \dots, j_d) = (\min\{i_1, j_1\}, \dots, \min\{i_d, j_d\}).$$

$\mathcal{C}(\setminus_\infty, \dots, \setminus_\Gamma)$ is a finite distributive lattice, and its zero and one are $(1, \dots, 1)$, (n_1, \dots, n_d) respectively.

Note that there is a total order \triangleleft on $\mathcal{C}(\setminus_\infty, \dots, \setminus_\Gamma)$ extending $<$, namely the lexicographic order, defined by $(i_1, \dots, i_d) \triangleleft (j_1, \dots, j_d)$ if and only if there exists $l < d$ such that $i_1 = j_1, \dots, i_l = j_l, i_{l+1} < j_{l+1}$. Also note that two elements $(i_1, \dots, i_d) \triangleleft (j_1, \dots, j_d)$ are non-comparable with respect to \leq if and only if there exists $1 < h \leq d$ such that $i_h > j_h$.

Sometimes we denote the elements of $\mathcal{C}(\setminus_\infty, \setminus_\epsilon, \dots, \setminus_\Gamma)$ by $x_{i_1 \dots i_d}$, with $1 \leq i_1 \leq n_1, \dots, 1 \leq i_d \leq n_d$.

2.7. The lattice of all subsets of the set $\{1, 2, \dots, n\}$ is denoted by $\mathcal{B}(\setminus)$, and called the *Boolean algebra of rank n* . Note that $\mathcal{B}(\setminus)$ is isomorphic to $[\mathcal{C}(\in)]^\setminus$.

One has the following (see [?]):

Theorem 2.8. *Any finite distributive lattice is isomorphic to a sublattice of a Boolean algebra of finite rank, and hence, in particular, to a sublattice of a finite chain product.*

3. THE IDEAL ASSOCIATED TO A DISTRIBUTIVE LATTICE

3.1. Let $T = (k^*)^m$ be the m -dimensional torus. Let M be the character group ($=\text{Hom}_{\text{alg. gp.}}(T, \mathbb{G}_{>})$) of T . Then M can be identified with $\mathbb{Z}^>$. Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be a subset of $\mathbb{Z}^>$. Consider the homomorphism

$$\pi : \mathbb{Z}^\times \rightarrow \mathbb{Z}^>, \quad \mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_n) \mapsto \mathbf{u}_1 \mathbf{a}_1 + \dots + \mathbf{u}_n \mathbf{a}_n.$$

Let $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$, $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_m)$, and

$$k[\mathbf{x}] = k[\mathbf{x}_1, \dots, \mathbf{x}_n], \quad k[\mathbf{t}^{\pm 1}] = k[\mathbf{t}_1, \dots, \mathbf{t}_m, \mathbf{t}_1^{-1}, \dots, \mathbf{t}_m^{-1}].$$

The map π induces a homomorphism of semigroup algebras

$$\hat{\pi} : k[\mathbf{x}] \rightarrow k[\mathbf{t}^{\pm 1}], \quad \mathbf{x}_i \mapsto \mathbf{t}^{\mathbf{a}_i}.$$

Definition 3.2. The kernel of $\hat{\pi}$ is denoted by $I_{\mathcal{A}}$ and called the *toric ideal* associated to \mathcal{A} .

Note that a toric ideal is prime.

Recall the following (see [?]).

Proposition 3.3. *The toric ideal $I_{\mathcal{A}}$ is spanned as a k -vector space by the set of binomials*

$$\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{Z}_+^\times \text{ with } \pi(\mathbf{u}) = \pi(\mathbf{v})\}. \quad (3)$$

3.4. **An important example.** Let us fix the integers $n_1, \dots, n_d > 1$, and let $n = \prod_{i=1}^d n_i$, $m = \sum_{i=1}^d n_i$. Let $\mathbf{e}_1^1, \dots, \mathbf{e}_{n_1}^1$ be the unit vectors in $\mathbb{Z}^{\times \leq}$, for $1 \leq l \leq d$. For $1 \leq \xi_1 \leq n_1, \dots, 1 \leq \xi_d \leq n_d$, define

$$\mathbf{a}_{\xi_1 \dots \xi_d} = \mathbf{e}_{\xi_1}^1 \oplus \dots \oplus \mathbf{e}_{\xi_d}^d \in \mathbb{Z}^{\times \#} \oplus \dots \oplus \mathbb{Z}^{\times}$$

and

$$\mathcal{A}_{\setminus \infty, \dots, \setminus \Gamma} = \{\mathbf{a}_{\xi_1 \dots \xi_d} \mid \mathbf{1} \leq \xi_1 \leq \mathbf{n}_1, \dots, \mathbf{1} \leq \xi_d \leq \mathbf{n}_d\}.$$

The corresponding map

$$\pi : \mathbb{Z}^{\times \# \dots \times} \rightarrow \mathbb{Z}^{\times \# + \dots + \times}$$

is defined as follows: for $1 \leq l \leq d$ and $1 \leq i_l \leq n_l$ fixed, the $(n_1 + \dots + n_{l-1} + i_l)$ -th coordinate of $\pi(\mathbf{u})$ is given by $\sum u_{\xi_1 \dots \xi_{l-1} \xi_l \xi_{l+1} \dots \xi_d}$, the sum being taken over the elements $(\xi_1, \dots, \xi_{l-1}, \xi_l, \xi_{l+1}, \dots, \xi_d)$ of $\mathcal{C}(\setminus \infty, \dots, \setminus \Gamma)$ with $\xi_l = i_l$. We call this subset the l -th *slice* of $\mathcal{C}(\setminus \infty, \dots, \setminus \Gamma)$ defined by i_l , and denote it by $\{\xi_l = i_l\}$. The components (or *entries*) of an element $\mathbf{u} \in \mathbb{Z}^{\times \# \dots \times}$ are indexed by the elements (i_1, \dots, i_d) of $\mathcal{C}(\setminus \infty, \dots, \setminus \Gamma)$. If $(j_1, \dots, j_d) \in \{\xi_l = i_l\}$, sometimes we also say that $u_{j_1 \dots j_d}$ itself belongs to the slice $\{\xi_l = i_l\}$.

The map π induces the map

$$\hat{\pi} : k[x_{11 \dots 1}, \dots, x_{\xi_1 \xi_2 \dots \xi_d}, \dots, x_{n_1 n_2 \dots n_d}] \rightarrow k[t_{11}, \dots, t_{1n_1}, \dots, t_{d1}, \dots, t_{dn_d}]$$

given by

$$x_{\xi_1 \dots \xi_d} \mapsto t_{1\xi_1} \dots t_{d\xi_d}, \text{ for } 1 \leq \xi_1 \leq n_1, \dots, 1 \leq \xi_d \leq n_d.$$

Definition 3.5. Given a finite lattice \mathcal{L} , the *ideal associated to \mathcal{L}* , denoted by $I(\mathcal{L})$, is the ideal of the polynomial ring $k[\mathcal{L}]$ generated by the set of binomials

$$\mathcal{G}_{\mathcal{L}} = \{\xi \dagger - (\xi \wedge \dagger)(\xi \vee \dagger) \mid \xi, \dagger \in \mathcal{L} \text{ non-comparable}\}.$$

By Theorem ??, a finite distributive lattice \mathcal{L} may be identified with a sublattice of a finite chain product lattice. Hence it inherits a total order extending the given partial order. In turn, this total order induces the reverse lexicographic order on the monomials in $k[\mathcal{L}]$, as in ??.

The following theorem shows that the ideal associated to a chain product lattice is toric.

Theorem 3.6. 1) We have $I(\mathcal{C}(\setminus_{\infty}, \dots, \setminus_{\Gamma})) = \mathcal{I}_{\mathcal{A}_{\setminus_{\infty}, \dots, \setminus_{\Gamma}}}$.

2) The set of binomials

$$\mathcal{G} = \{\xi \dagger - (\xi \wedge \dagger)(\xi \vee \dagger) \mid \xi, \dagger \in \mathcal{C}(\setminus_{\infty}, \dots, \setminus_{\Gamma}) \text{ non-comparable}\}$$

is a Gröbner basis for $I(\mathcal{C}(\setminus_{\infty}, \dots, \setminus_{\Gamma}))$ with respect to the reverse lexicographic order.

Proof. Let $\mathcal{C} = \mathcal{C}(\setminus_{\infty}, \dots, \setminus_{\Gamma})$ and $\mathcal{A} = \mathcal{A}_{\setminus_{\infty}, \dots, \setminus_{\Gamma}}$. Let $f \in I_{\mathcal{A}}$; by Proposition ??, there exist $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{Z}_+^{\times}$ with $\pi(\mathbf{u}_i) = \pi(\mathbf{v}_i)$, and $c_i \in k^*$, $1 \leq i \leq s$ such that

$$f = \sum_{i=1}^s c_i (\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) \quad (4)$$

for some $s \geq 1$, with the property that s is the smallest integer ≥ 1 such that f can be expressed as a linear combination of s binomials in the set (?). Now we rewrite f as

$$f = \sum_{i=1}^s a_i \mathbf{x}^{\mathbf{u}_i} + \sum_{i=1}^s \mathbf{b}_i \mathbf{x}^{\mathbf{v}_i}, \quad \mathbf{a}_i, \mathbf{b}_i \in \mathbf{k}.$$

Then none of the coefficients $a_1, \dots, a_s, b_1, \dots, b_s$ is zero. Indeed, suppose that $a_i = 0$ for some $1 \leq i \leq s$. This implies that there exists $j \in \{1, \dots, s\}$, $j \neq i$ such that either $c_j = c_i$ and $\mathbf{v}_j = \mathbf{u}_i$, or $c_j = -c_i$ and $\mathbf{u}_j = \mathbf{u}_i$. In the first case we have

$$c_i (\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) + c_j (\mathbf{x}^{\mathbf{u}_j} - \mathbf{x}^{\mathbf{v}_j}) = c_i (\mathbf{x}^{\mathbf{u}_j} - \mathbf{x}^{\mathbf{v}_i}), \quad \pi(\mathbf{u}_j) = \pi(\mathbf{v}_i). \quad (5)$$

In the second case we have

$$c_i(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) + \mathbf{c}_j(\mathbf{x}^{\mathbf{u}_j} - \mathbf{x}^{\mathbf{v}_j}) = \mathbf{c}_i(\mathbf{x}^{\mathbf{v}_j} - \mathbf{x}^{\mathbf{v}_i}), \quad \pi(\mathbf{v}_j) = \pi(\mathbf{v}_i). \quad (6)$$

But (??), (??) and (??) imply that f can be written as a linear combination of $s - 1$ binomials in the set (??), contradicting the choice of s . Thus $a_i \neq 0$, $1 \leq i \leq s$, and similarly $b_i \neq 0$, $1 \leq i \leq s$. This shows that $\text{in}(f) = \text{in}(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i})$ for some $1 \leq i \leq s$. Let us write

$$\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i} = \mathbf{x}^{\mathbf{w}}(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}),$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{Z}_+^\times$, with $\pi(\mathbf{u}) = \pi(\mathbf{v})$ and $\text{supp}(\mathbf{u}) \cap \text{supp}(\mathbf{v}) = \emptyset$. Let us suppose that $\mathbf{x}^{\mathbf{u}} \succeq_{\text{rlex}} \mathbf{x}^{\mathbf{v}}$, i.e. $\text{in}(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}) = \mathbf{x}^{\mathbf{u}}$ and $\text{in}(f) = \text{in}(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) = \mathbf{x}^{\mathbf{u}_i}$. Let $x_{i_1 \dots i_d}$ be the smallest variable appearing in $\mathbf{x}^{\mathbf{u}}$, i.e. (i_1, \dots, i_d) is the smallest element of $\text{supp}(\mathbf{u})$ with respect to \triangleleft . Then $\mathbf{x}^{\mathbf{v}}$ contains a variable $x_{k_1 \dots k_d}$, with $(k_1, \dots, k_d) \triangleleft (i_1, \dots, i_d)$. Since $\pi(\mathbf{u}) = \pi(\mathbf{v})$, the sum of the entries in every slice is the same for both \mathbf{u} and \mathbf{v} . In particular, since all the entries of \mathbf{u} in the slices $\{\xi_1 = i\}$, with $1 \leq i < i_1$, are 0 (by the choice of (i_1, \dots, i_d)), all the entries of \mathbf{v} in these slices must also be 0. This implies that $(k_1, \dots, k_d) \in \{\xi_1 = i_1\}$. Let $1 < h \leq n_1$ such that $k_1 = i_1, \dots, k_{h-1} = i_{h-1}, k_h < i_h$. Then the sum of the elements of \mathbf{v} in the slice $\{\xi_h = k_h\}$ is nonzero, which implies that $\{\xi_h = k_h\} \cap \text{supp}(\mathbf{u}) \neq \emptyset$. Let (j_1, \dots, j_d) be an element in this intersection. We have $(i_1, \dots, i_d) \triangleleft (j_1, \dots, j_d)$ (by the definition of (i_1, \dots, i_d)), and since $i_h > k_h = j_h$, we conclude that (i_1, i_2, \dots, i_d) and (j_1, \dots, j_d) are non-comparable. Thus we obtain that $\mathbf{x}^{\mathbf{u}}$ is divisible by $x_{i_1 \dots i_d} x_{j_1 \dots j_d}$. Hence $\mathbf{x}^{\mathbf{u}_i}$ is also divisible by $x_{i_1 \dots i_d} x_{j_1 \dots j_d}$. Therefore $\text{in}(f)$ is divisible by the initial term of an element of the set

$$\mathcal{G} = \{\xi \dagger - (\xi \wedge \dagger)(\xi \vee \dagger) \mid \xi, \dagger \in \mathcal{C} \text{ non-comparable}\}$$

of generators of the ideal $I(\mathcal{C})$. Since $\mathcal{G} \subset \mathcal{I}_{\mathcal{A}}$, it follows that \mathcal{G} is a Gröbner basis for $I_{\mathcal{A}}$. In particular it is a set of generators for this ideal. Thus \mathcal{G} generates both $I(\mathcal{C})$ and $I_{\mathcal{A}}$, which implies the equality of the two ideals. \square

Theorem 3.7. *Let \mathcal{L} be a finite distributive lattice. Then*

- 1) *The ideal $I(\mathcal{L})$ is toric (and hence prime).*
- 2) *The set of binomials*

$$\mathcal{G}_{\mathcal{L}} = \{\xi \dagger - (\xi \wedge \dagger)(\xi \vee \dagger) \mid \xi, \dagger \in \mathcal{L} \text{ non-comparable}\}$$

is a Gröbner basis for $I(\mathcal{L})$ with respect to the reverse lexicographic order.

Proof. By Theorem ??, we may assume that \mathcal{L} is a sublattice of $\mathcal{C}(\backslash_\infty, \dots, \backslash_\Gamma)$, for some $n_1, \dots, n_d \geq 1$. Let us denote $\mathcal{C} = \mathcal{C}(\backslash_\infty, \dots, \backslash_\Gamma)$, $\mathcal{A} = \mathcal{A}_{\backslash_\infty, \dots, \backslash_\Gamma}$ and $\mathcal{G} = \mathcal{G}_{\mathcal{C}}$. Note that $\mathcal{G}_{\mathcal{L}}$ is the subset of \mathcal{G} consisting of all binomials in \mathcal{G} involving only the variables from \mathcal{L} . Let us denote

$$\mathcal{G}_{\mathcal{L}} = \{\{\infty, \dots, \{\nabla\}\}, \quad \mathcal{G} \setminus \mathcal{G}_{\mathcal{L}} = \{\{\infty, \dots, \}\}_f\}$$

Let $g_i = xy - (x \wedge y)(x \vee y)$, with $x, y \in \mathcal{C}$ non-comparable, $1 \leq i \leq s$; then at least one of x and y does not belong to \mathcal{L} (\mathcal{L} being a sublattice of \mathcal{C} , $x, y \in \mathcal{L}$ would imply $x \wedge y, x \vee y \in \mathcal{L}$, so g_i would involve only variables from \mathcal{L}).

Let $\mathcal{A}_{\mathcal{L}} \subset \mathcal{A}$ be given by the elements in \mathcal{A} indexed by the elements of \mathcal{L} , and let f be an element of

$$I_{\mathcal{A}_{\mathcal{L}}} = \ker \left(\hat{\pi} \Big|_{k[\mathcal{L}]} \right) = (\ker \hat{\pi}) \cap k[\mathcal{L}] = \mathcal{I}_{\mathcal{A}} \cap \|\mathcal{L}\|.$$

In the course of the proof of Theorem ??, we saw that $\text{in}(f)$ is divisible by the initial term of a binomial in \mathcal{G} , and since $f \in k[\mathcal{L}]$, this binomial must be one of the f_i 's, i.e. an element of $\mathcal{G}_{\mathcal{L}}$. Since $\mathcal{G}_{\mathcal{L}} \subset \mathcal{I}_{\mathcal{A}_{\mathcal{L}}}$, it follows that $\mathcal{G}_{\mathcal{L}}$ is a Gröbner basis for $I_{\mathcal{A}_{\mathcal{L}}}$, hence $\mathcal{G}_{\mathcal{L}}$ generates $I_{\mathcal{A}_{\mathcal{L}}}$. Therefore $I(\mathcal{L}) = \mathcal{I}_{\mathcal{A}_{\mathcal{L}}}$, and the stated assertions follow now. \square

4. VARIETIES DEFINED BY BINOMIALS

Let k be an algebraically closed field of arbitrary characteristic. Consider an integer $m \geq 1$, and the m -dimensional torus $T_m = (k^*)^m$

Definition 4.1. (cf. [?]) An equivariant affine embedding of T_m is an affine variety X containing T_m as an open subset, equipped with an action of T_m on X

$$T_m \times X \rightarrow X$$

extending the action $T_m \times T_m \rightarrow T_m$ given by the translations in T_m .

Definition 4.2. (cf. [?], [?]) An equivariant affine embedding X of a torus is called an *affine toric variety* if it is normal.

4.3. Let $N \geq 1$, and let X be an affine variety in \mathbb{A}^N , not contained in any of the coordinate hyperplanes $\{x_i = 0\}$. Further, let X be irreducible, and let its defining prime ideal $I(X)$ be generated by l binomials

$$x_1^{a_{i1}} \dots x_N^{a_{iN}} - \lambda_i x_1^{b_{i1}} \dots x_N^{b_{iN}} \quad , \quad 1 \leq i \leq l \quad . \quad (*)$$

Consider the natural action of the torus $T_N = (k^*)^N$ on \mathbb{A}^N ,

$$(t_1, \dots, t_N) \cdot (a_1, \dots, a_N) = (t_1 a_1, \dots, t_N a_N).$$

Let $X(T_N) = \text{Hom}(T_N, \mathbb{G}_{>})$ be the character group of T_N , and let $\varepsilon_i \in X(T_N)$ be the character

$$\varepsilon_i(t_1, \dots, t_N) = t_i \quad , \quad 1 \leq i \leq N.$$

For $1 \leq i \leq l$, let

$$\varphi_i = \sum_{t=1}^N (a_{it} - b_{it}) \varepsilon_t \quad .$$

Set $T = \bigcap_{i=1}^l \ker \varphi_i$, and $X^\circ = \{(x_1, \dots, x_N) \in X \mid x_i \neq 0 \text{ for all } 1 \leq i \leq N\}$.

Proposition 4.4. *Let notations be as above.*

- (1) *There is a canonical action of T on X .*
- (2) *X° is T -stable. Further, the action of T on X° is simple and transitive.*
- (3) *T is a subtorus of T_N , and X is an equivariant affine embedding of T .*

Proof. (1) We consider the (obvious) action of T on \mathbb{A}^N . Let $(x_1, \dots, x_N) \in X$, $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_N) \in \mathbf{T}$, and $(y_1, \dots, y_N) = \mathbf{t} \cdot (\mathbf{x}_1, \dots, \mathbf{x}_N) = (\mathbf{t}_1 \mathbf{x}_1, \dots, \mathbf{t}_N \mathbf{x}_N)$. Using the fact that (x_1, \dots, x_N) satisfies $(*)$, we obtain

$$y_1^{a_{i1}} \dots y_N^{a_{iN}} = t_1^{a_{i1}} \dots t_N^{a_{iN}} x_1^{a_{i1}} \dots x_N^{a_{iN}} = \lambda_i t_1^{b_{i1}} \dots t_N^{b_{iN}} x_1^{b_{i1}} \dots x_N^{b_{iN}} = \lambda_i y_1^{b_{i1}} \dots y_N^{b_{iN}} ,$$

for all $1 \leq i \leq l$, i.e. $(y_1, \dots, y_N) \in X$. Hence $\mathbf{t} \cdot (\mathbf{a}_1, \dots, \mathbf{a}_N) \in \mathbf{X}$ for all $\mathbf{t} \in \mathbf{X}$.

(2) Let $x = (x_1, \dots, x_N) \in X^\circ$, and $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_N) \in \mathbf{T}$. Then, clearly $\mathbf{t} \cdot (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbf{X}^\circ$. Considering x as a point in \mathbb{A}^N , the isotropy subgroup in T_N at x is $\{\text{id}\}$. Hence the isotropy subgroup in T at x is also $\{\text{id}\}$. Thus the action of T on X° is simple.

Let $(x_1, \dots, x_N), (x'_1, \dots, x'_N) \in X^\circ$. Set $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_N)$, where $t_i = x_i/x'_i$. Then, clearly $\mathbf{t} \in \mathbf{T}$. Thus $(x_1, \dots, x_N) = \mathbf{t} \cdot (\mathbf{x}'_1, \dots, \mathbf{x}'_N)$. Hence the action of T on X° is simple and transitive.

(3) Now, fixing a point $x \in X^\circ$, we obtain from (2) that the orbit map $\mathbf{t} \mapsto \mathbf{t} \cdot \mathbf{x}$ is in fact an isomorphism of T onto X° . Also, since X is not contained in any of the coordinate hyperplanes, the open set $X_i = \{(x_1, \dots, x_N) \in X \mid x_i \neq 0\}$ is nonempty for all $1 \leq i \leq N$. The irreducibility of X implies that the sets X_i , $1 \leq i \leq N$, are open dense in X , and hence their intersection

$$X^\circ = \bigcap_{i=1}^N X_i = \{(x_1, \dots, x_N) \in X \mid x_i \neq 0 \text{ for any } i\}$$

is an open dense set in X , and thus X° is irreducible. This implies that T is irreducible (and hence connected). Thus T is a subtorus of T_N .

The assertion that X is an equivariant affine embedding of T follows from (1) and (2). \square

Remark 4.5. One can see that the ideal $I(X)$ is a toric ideal in the sense of Definition ??; more precisely, $I(X) = I_{\mathcal{A}}$, with $m = \dim T$, $n = N$, $\mathcal{A} = \{\rho, \infty \leq \} \leq \mathcal{N}\}$, where $\rho_i = \varepsilon_i|_T$ (here $k[T]$ is identified with $k[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$, and the character group $X(T)$ with \mathbb{Z}^{\geq}).

5. SOME GENERAL RESULTS ON THE VARIETY ASSOCIATED TO A FINITE DISTRIBUTIVE LATTICE

We first recall some basic definitions on finite partially ordered sets. A partially ordered set is also called a poset.

Definition 5.1. A finite poset P is called *bounded* if it has a unique maximal, and a unique minimal element, denoted $\hat{1}$ and $\hat{0}$ respectively.

Definition 5.2. A totally ordered subset C of a finite poset P is called a *chain*, and the number $\#C - 1$ is called the *length* of the chain.

Definition 5.3. A bounded poset P is said to be *graded* if all maximal chains have the same length (note that $\hat{1}$ and $\hat{0}$ belong to any maximal chain).

Definition 5.4. Let P be a graded poset. The length of a maximal chain in P is called the *rank* of P .

Definition 5.5. Let P be a graded poset. For $\lambda, \mu \in P$ with $\lambda \geq \mu$, the graded poset $\{\tau \in P \mid \mu \leq \tau \leq \lambda\}$ is called the *interval from μ to λ* , and denoted by $[\mu, \lambda]$. The rank of $[\mu, \lambda]$ is denoted by $l_{\mu}(\lambda)$; if $\mu = \hat{0}$, then we denote $l_{\mu}(\lambda)$ by just $l(\lambda)$.

Definition 5.6. Let P be a graded poset, and $\lambda, \mu \in P$, with $\lambda \geq \mu$. The ordered pair (λ, μ) is called a *cover* (and we also say that λ *covers* μ) if $l_{\mu}(\lambda) = 1$.

Definition 5.7. An element z of a lattice \mathcal{L} is called *join-irreducible* (respectively *meet-irreducible*) if $z = x \vee y$ (respectively $z = x \wedge y$) implies $z = x$ or $z = y$. The set of join-irreducible (respectively meet-irreducible) elements of \mathcal{L} is denoted by $J_{\mathcal{L}}$ (respectively $M_{\mathcal{L}}$), or just by J (respectively M) if no confusion is possible.

Definition 5.8. The set $J_{\mathcal{L}} \cap M_{\mathcal{L}}$ of join and meet-irreducible elements is denoted by $JM_{\mathcal{L}}$, or just JM if no confusion is possible.

Definition 5.9. A subset I of a poset P is called an *ideal* of P if for all $x, y \in P$,

$$x \in I \text{ and } y \leq x \text{ imply } y \in I.$$

Theorem 5.10. (*Birkhoff*) *Let \mathcal{L} be a distributive lattice with 0 , and P the poset of its nonzero join-irreducible elements. Then \mathcal{L} is isomorphic to the lattice of finite ideals of P , by means of the lattice isomorphism*

$$\alpha \mapsto I_\alpha = \{\tau \in P \mid \tau \leq \alpha\}, \quad \alpha \in \mathcal{L}.$$

Definition 5.11. A quadruple of the form $(\tau, \phi, \tau \vee \phi, \tau \wedge \phi)$, with $\tau, \phi \in \mathcal{L}$ non-comparable is called a *diamond*, and is denoted by $D(\tau, \phi, \tau \vee \phi, \tau \wedge \phi)$.

Lemma 5.12. *With the notations as above, we have*

- (a) $J = \{\tau \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\tau, \lambda)\}$.
- (b) $M = \{\tau \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\lambda, \tau)\}$.

Proof. In order to prove part (a), let us denote

$$Z = \{\lambda \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\lambda, \rho)\}$$

Clearly, $J \subset Z$. Let $\lambda \in Z$, and assume $\lambda \notin J$. Assumption implies that there exists a diamond $D(\tau, \phi, \lambda, \mu)$. In particular, this implies $\tau \leq \lambda, \phi \leq \lambda$. Now \mathcal{L} being graded, there exist α, β such that $\tau \leq \alpha, \phi \leq \beta$, and $(\lambda, \alpha), (\lambda, \beta)$ are covers. Now, $\tau \leq \alpha, \phi \leq \beta$, and $\lambda = \alpha \vee \beta$ imply $\alpha \neq \beta$. But this would imply that λ covers two distinct elements, contradicting the hypothesis that $\lambda \in Z$. Hence the assumption that $\lambda \notin J$ is wrong, which shows (a).

The proof of (b) is similar. \square

5.13. Let $X(\mathcal{L})$ be the affine variety of the zeroes in k^N of $I(\mathcal{L})$ (note that $X(\mathcal{L})$ is irreducible, in view of Theorem ??). Then $X(\mathcal{L})$ is a variety defined by binomials, and we follow the notations in Section ??. We have $N = \#\mathcal{L}$, $I(X(\mathcal{L})) = \mathcal{I}(\mathcal{L})$. Let $\mathcal{I} = \{(\tau, \phi, \tau \vee \phi, \tau \wedge \phi) \mid (\tau, \phi) \in \mathcal{Q}\}$, where

$$\mathcal{Q} = \{(\tau, \phi) \mid \tau, \phi \in \mathcal{L} \text{ non-comparable}\}.$$

In view of Proposition ??, $\dim X(\mathcal{L}) = \dim \mathcal{T}$, and we now compute the dimension of T .

Let $\pi : X(T_N) \rightarrow X(T)$ be the canonical map, given by restriction, and for $\chi \in X(T_N)$, denote $\pi(\chi)$ by $\bar{\chi}$. Let us fix a \mathbb{Z} -basis $\{\chi_\tau \mid \tau \in \mathcal{L}\}$ for $X(T_N)$. For a diamond $D = (\tau, \phi, \tau \vee \phi, \tau \wedge \phi) \in \mathcal{I}$, let $\chi_D = \chi_{\tau \vee \phi} + \chi_{\tau \wedge \phi} - \chi_\tau - \chi_\phi$.

Lemma 5.14. *We have*

- (1) $X(T) \simeq X(T_N) / \ker \pi$.
- (2) $\ker \pi$ is generated by $\{\chi_D \mid D \in \mathcal{I}\}$.

Proof. The canonical map π is, in fact, surjective, since T is a subtorus of T_N . Now (1) follows from this. The assertion (2) follows from the definition of T . \square

5.15. For $\alpha \in \mathcal{L}$, let I_α be the ideal corresponding to α under the isomorphism in Theorem ???. Let

$$\psi_\alpha = \sum_{\theta \in I_\alpha} \chi_\theta.$$

Lemma 5.16. *The set $\{\bar{\psi}_a \mid a \in \mathcal{L}\}$ generates $X(T)$ as a \mathbb{Z} -module.*

Proof. Consider the homomorphism

$$\theta : X(T_N) \rightarrow X(T) \quad , \quad \chi_a \mapsto \bar{\psi}_a.$$

(note that $\{\chi_a \mid a \in \mathcal{L}\}$ is a \mathbb{Z} -basis for $X(T_N)$). For a diamond $D = (\tau, \phi, \mu, \lambda) \in \mathcal{I}$, we have

$$\bar{\psi}_\tau + \bar{\psi}_\phi = \bar{\psi}_\mu + \bar{\psi}_\lambda, \quad (*)$$

and hence $\chi_D \in \ker \theta$. Conversely, it is clear that any relation among $\bar{\psi}_a$'s is of the form (*). Hence $\ker \theta = \ker \pi$, and $X(T) \simeq X(T_N) / \ker \theta$, (cf. Lemma ??). In particular, this implies that θ is onto, and the result follows, since $\{\chi_a \mid a \in \mathcal{L}\}$ generates $X(T_N)$ as a \mathbb{Z} -module. \square

Proposition 5.17. *The set $\{\bar{\chi}_\tau \mid \tau \in J\}$ is a \mathbb{Z} -basis for $X(T)$.*

Proof. By Lemma ??, $\{\bar{\psi}_a \mid a \in \mathcal{L}\}$ generates the \mathbb{Z} -module $X(T)$. Now

$$\bar{\psi}_a = \sum_{\theta \in I_a} \bar{\chi}_\theta = \sum_{\{\theta \in J_\mathcal{L}, \theta \leq a\}} \bar{\chi}_\theta \quad (\dagger)$$

(cf. ??). Hence $\{\bar{\chi}_\tau \mid \tau \in J\}$ generates the \mathbb{Z} -module $X(T)$. Also it is clear that no proper subset of $\{\bar{\chi}_\tau \mid \tau \in J\}$ generates $X(T)$. The result now follows. \square

Now, since $\dim X(\mathcal{L}) = \dim \mathcal{T}$, we obtain

Theorem 5.18. *The dimension of $X(\mathcal{L})$ is equal to $\#J_\mathcal{L}$.*

Definition 5.19. Let \mathcal{L} be a finite distributive lattice. We call the cardinality of $J_\mathcal{L}$ the *dimension* of \mathcal{L} , and we denote it by $\dim \mathcal{L}$. If \mathcal{L}' is a sublattice of \mathcal{L} , then the *codimension* of \mathcal{L}' in \mathcal{L} is defined as $\dim \mathcal{L} - \dim \mathcal{L}'$.

Lemma 5.20. *Let $P = (P_\theta)_{\theta \in \mathcal{L}} \in X(\mathcal{L})$ be such that $P_\tau \neq 0$ for any $\tau \in J$. Then $P_\theta \neq 0$ for any $\theta \in \mathcal{L}$.*

Proof. (by induction) Let $\theta \in \mathcal{L}$. If $\theta \in J_\mathcal{L}$, there is nothing to check. Let then $\theta \in \mathcal{L} \setminus J_\mathcal{L}$.

Let θ be a minimal element of $\mathcal{L} \setminus J_\mathcal{L}$. This implies that every $\tau \in \mathcal{L}$ such that $\tau < \theta$ belongs to $J_\mathcal{L}$. The fact that $\theta \in \mathcal{L} \setminus J_\mathcal{L}$ implies that there are at least two elements θ_1, θ_2 of \mathcal{L} which are covered by θ . Note that θ_1, θ_2 are not comparable. We have $\theta_1 \vee \theta_2 = \theta$. Let $\mu = \theta_1 \wedge \theta_2$.

We have $P_\theta P_\mu = P_{\theta_1} P_{\theta_2}$. Now $P_{\theta_1} \neq 0$, $P_{\theta_2} \neq 0$, since $\theta_1, \theta_2 \in J_{\mathcal{L}}$. Hence we obtain that $P_\theta \neq 0$.

Let now ϕ be any element of $\mathcal{L} \setminus J_{\mathcal{L}}$. Assume, by induction, that $P_\tau \neq 0$ for any $\tau < \phi$. Since $\phi \notin J_{\mathcal{L}}$, there are at least two elements ϕ_1, ϕ_2 of \mathcal{L} which are covered by ϕ . We have $\phi_1 \vee \phi_2 = \phi$, $P_\phi P_\delta = P_{\phi_1} P_{\phi_2}$, where $\delta = \phi_1 \wedge \phi_2$. Also $P_{\phi_1} \neq 0$, $P_{\phi_2} \neq 0$ (since ϕ_1, ϕ_2 are both $< \phi$). Hence we obtain $P_\phi \neq 0$. \square

Definition 5.21. (cf. [?]) A sublattice \mathcal{L}' of \mathcal{L} is called an *embedded sublattice* of \mathcal{L} if

$$\tau, \phi \in \mathcal{L}, \quad \tau \vee \phi, \tau \wedge \phi \in \mathcal{L}' \quad \Rightarrow \quad \tau, \phi \in \mathcal{L}'.$$

Given a sublattice \mathcal{L}' of \mathcal{L} , let us consider the variety $X(\mathcal{L}')$, and consider the canonical embedding $X(\mathcal{L}') \hookrightarrow \mathbb{A}(\mathcal{L}') \hookrightarrow \mathbb{A}(\mathcal{L})$ (here $\mathbb{A}(\mathcal{L}') = \mathbb{A}^{\#\mathcal{L}'}$, $\mathbb{A}(\mathcal{L}) = \mathbb{A}^{\#\mathcal{L}}$).

Proposition 5.22. $X(\mathcal{L}')$ is a subvariety of $X(\mathcal{L})$ if and only if \mathcal{L}' is an embedded sublattice of \mathcal{L} .

Proof. Under the embedding $X(\mathcal{L}') \hookrightarrow \mathbb{A}(\mathcal{L})$, $X(\mathcal{L}')$ can be identified with

$$\{(x_\tau)_{\tau \in \mathcal{L}} \in \mathbb{A}(\mathcal{L}) \mid \xi_\tau = \iota \text{ if } \tau \notin \mathcal{L}', \xi_\tau \xi_\phi = \xi_{\tau \vee \phi} \xi_{\tau \wedge \phi} \text{ for } \tau, \phi \in \mathcal{L}' \text{ noncomparable}\}.$$

Let η' be the generic point of $X(\mathcal{L}')$. We have $X(\mathcal{L}') \subset X(\mathcal{L})$ if and only if $\eta' \in X(\mathcal{L})$.

Assume that $\eta' \in X(\mathcal{L})$. Let τ, ϕ be two noncomparable elements of \mathcal{L} such that $\tau \vee \phi, \tau \wedge \phi$ are both in \mathcal{L}' . We have to show that $\tau, \phi \in \mathcal{L}'$. If possible, let $\tau \notin \mathcal{L}'$. This implies $\eta'_\tau = 0$. Hence either $\eta'_{\tau \vee \phi} = 0$, or $\eta'_{\tau \wedge \phi} = 0$, since $\eta' \in X(\mathcal{L})$. But this is not possible (note that $\tau \vee \phi, \tau \wedge \phi$ are in \mathcal{L}' , and hence $\eta'_{\tau \vee \phi}$ and $\eta'_{\tau \wedge \phi}$ are both nonzero).

Assume now that \mathcal{L}' is an embedded sublattice. We have to show that $\eta' \in X(\mathcal{L})$. Let τ, ϕ be two noncomparable elements of \mathcal{L} . The fact that \mathcal{L}' is a sublattice implies that if $\eta'_{\tau \vee \phi}$ or $\eta'_{\tau \wedge \phi}$ is zero, then either η'_τ , or η'_ϕ is zero. Also, the fact that \mathcal{L}' is an embedded sublattice implies that if η'_τ or η'_ϕ is zero, then either $\eta'_{\tau \vee \phi}$ or $\eta'_{\tau \wedge \phi}$ is zero. Further, when $\tau, \phi, \tau \vee \phi, \tau \wedge \phi \in \mathcal{L}'$,

$$\eta'_\tau \eta'_\phi = \eta'_{\tau \vee \phi} \eta'_{\tau \wedge \phi}.$$

Thus η' satisfies the defining equations of $X(\mathcal{L})$, and hence $\eta' \in X(\mathcal{L})$. \square

6. A STANDARD MONOMIAL BASIS FOR $R(\mathcal{L})$

6.1. Let $R(\mathcal{L}) = \mathbb{K}[\mathcal{L}]/\mathcal{I}(\mathcal{L})$, $k[\mathcal{L}]$ being as in Section ???. Note that $R(\mathcal{L})$ is a domain (cf. Theorem ???). Let $X(\mathcal{L}) = \text{Spec} \mathcal{R}(\mathcal{L})$. For

$\theta \in \mathcal{L}$, let us consider the sublattice $\mathcal{L}_\theta = \{\lambda \in \mathcal{L} \mid \lambda \leq \theta\}$ and the ring $R_\theta = R_{\mathcal{L}}/\langle x_\lambda \mid \lambda \in \mathcal{L}, \lambda \not\leq \theta \rangle \cong \mathcal{R}(\mathcal{L}_\theta)$ and denote $\text{Spec}R_\theta$ by $X(\mathcal{L}_\theta)$, or just X_θ . Note that \mathcal{L}_θ is an embedded sublattice (cf. Definition ??). In the sequel, for $\tau \in \mathcal{L}$, we shall denote \bar{x}_τ (in $R(\mathcal{L})$) by just x_τ ; similarly, the restriction $x_\tau|_{X_\theta}$ will be also denoted by just x_τ .

6.2. We take a total order \succ on \mathcal{L} extending the partial order $>$, and a monomial of degree r in $R(\mathcal{L})$ will be written as $x_{\tau_1} \dots x_{\tau_r}$, with $\tau_1 \succeq \dots \succeq \tau_r$.

Definition 6.3. A monomial $x_{\tau_1} \dots x_{\tau_r}$ of degree r is said to be *standard on $X(\mathcal{L})$* , if $\tau_1 \geq \dots \geq \tau_r$. Such a monomial is said to be *standard on X_θ* if $\theta \geq \tau_1 \geq \dots \geq \tau_r$.

Proposition 6.4. *Standard monomials on X_θ are linearly independent in R_θ .*

Proof. R_θ being graded, it suffices to prove the linear independence of standard monomials of a given degree, say r . We shall prove this by induction on $l(\theta)$ (cf. Definition ??), and r . If $l(\theta) = 0$, or $r = 1$, the result is clear. Let then $l(\theta) \geq 1$, $r > 1$, and let

$$\sum_{i \in I} a_i F_i = 0, \quad a_i \in k^* \quad (*)$$

be a linear relation, where $F_i = x_{\tau_{i1}} \dots x_{\tau_{ir}}$, $\tau_{i1} \geq \dots \geq \tau_{ir}$. If $\tau_{i1} < \theta$ for some i , then restricting $(*)$ to $X_{\tau_{i1}}$, we obtain (by induction hypothesis) that $a_j = 0$ for all j with $\tau_{j1} \leq \tau_{i1}$ (note that $F_j|_{X_{\tau_{i1}}}$ is either identically zero, or it remains standard on $X_{\tau_{i1}}$). Thus we may suppose that $\tau_{i1} = \theta$ for all $i \in I$. Now x_θ can be cancelled out (since R_θ is a domain), and the result follows by induction on r . \square

Proposition 6.5. *Any monomial in R_θ is standard.*

Proof. Let $F = x_{\tau_1} \dots x_{\tau_r}$. If there is an i such that $\tau_i \not\leq \tau_{i+1}$, this implies τ_i, τ_{i+1} are non-comparable. Then, denoting $\tau_i \vee \tau_{i+1}, \tau_i \wedge \tau_{i+1}$ by λ, μ respectively, we have $F = x_{\tau_1} \dots x_\lambda x_\mu \dots x_{\tau_r}$, and the new expression of F is lexicographically greater than the previous one, since $\lambda > \tau_i$. Continuing this process, after a finite number of steps we arrive at a standard monomial expression of F (since \mathcal{L} is finite). \square

Combining Propositions ?? and ??, we obtain

Theorem 6.6. *Standard monomials on X_θ form a basis of R_θ .*

7. COHEN-MACAULAYNESS OF $R(\mathcal{L})$

In this section we prove the Cohen-Macaulayness of $R(\mathcal{L})$ using deformation techniques. Let $S(\mathcal{L})$ denote the Stanley-Reisner of \mathcal{L} , namely $k[\mathcal{L}]/\langle \{x_\alpha x_\beta \mid (\alpha, \beta) \in \mathcal{Q}\} \rangle$ (recall that $\mathcal{Q} = \{(\alpha, \beta) \mid \alpha, \beta \in \mathcal{L} \text{ noncomparable}\}$).

Recall the following (cf. [?]):

Theorem 7.1. *The ring $S(\mathcal{L})$ is Cohen-Macaulay.*

We now construct a flat family over $k[t]$ whose general fiber is $R(\mathcal{L})$ and special fiber is $S(\mathcal{L})$. This construction is done in the same spirit as in [?].

7.2. We first assign positive integers n_τ , $\tau \in \mathcal{L}$, in such a way that $\tau > \tau'$, then $n_\tau > n_{\tau'}$ (for example we may take $n_\tau = l(\tau)$). We choose an integer N , and set $N_\tau = N^{l(\tau)}$. Then, since \mathcal{L} is finite, we can choose N sufficiently large so that for any diamond $D(\tau, \phi, \tau \vee \phi, \tau \wedge \phi)$,

$$N_{\tau \vee \phi} + N_{\tau \wedge \phi} > N_\tau + N_\phi.$$

Theorem 7.3. *There exists a flat family over $k[t]$ whose special fiber is $S(\mathcal{L})$ and general fiber is $R(\mathcal{L})$.*

Proof. For $\tau, \phi \in \mathcal{Q}$, let $f_{\tau, \phi}$ be the element in $k[\mathcal{L}]$ given by

$$f_{\tau, \phi} = x_\tau x_\phi - x_{\tau \vee \phi} x_{\tau \wedge \phi}.$$

Then $I(\mathcal{L}) = \langle \{f_{\tau, \phi} \mid (\tau, \phi) \in \mathcal{Q}\} \rangle$. Let us denote $R(\mathcal{L})$, $S(\mathcal{L})$, $k[\mathcal{L}]$, by R , S , P respectively. Let $A = k[t]$, and $P_A = A[x_\alpha, \alpha \in \mathcal{L}]$. For $(\tau, \phi) \in \mathcal{Q}$, we define the element $f_{\tau, \phi, t}$ in P_A as

$$f_{\tau, \phi, t} = x_\tau x_\phi - x_{\tau \vee \phi} x_{\tau \wedge \phi} t^{N_{\tau \vee \phi} + N_{\tau \wedge \phi} - N_\tau - N_\phi}.$$

(note that $f_{\tau, \phi, t}$ is well defined, in view of the choice of N_α 's).

Let \mathcal{I} be the ideal in P_A generated by $\{f_{\tau, \phi, t} \mid (\tau, \phi) \in \mathcal{Q}\}$, and $\mathcal{R} = \mathcal{P}_A / \mathcal{I}$.

Claim . (a) \mathcal{R} is $k[t]$ -free.

(b) $\mathcal{R} \otimes_{\llbracket \sqcup \rrbracket} \llbracket \sqcup, \sqcup^{-\infty} \rrbracket \simeq \mathcal{R}[\sqcup, \sqcup^{-\infty}]$.

(c) $\mathcal{R} \otimes_{\llbracket \sqcup \rrbracket} \llbracket \sqcup \rrbracket / (\sqcup) \simeq \mathcal{S}$.

Proof. We have

$$\mathcal{R} \otimes_{\llbracket \sqcup \rrbracket} \llbracket \sqcup \rrbracket / (\sqcup) = \mathcal{P}_A / \langle \mathcal{I} + (\sqcup) \rangle = \mathcal{S}$$

This proves (c). Let $B = k[t, t^{-1}]$, and $P_B = B[x_\alpha, \alpha \in \mathcal{L}]$. Let \tilde{I} (resp. $\tilde{\mathcal{I}}$) be the ideal in P_B generated by $\{f_{\tau, \phi} \mid (\tau, \phi) \in \mathcal{Q}\}$ (resp. $\{f_{\tau, \phi, t} \mid (\tau, \phi) \in \mathcal{Q}\}$). We have

$$P_B / \tilde{I} \simeq R[t, t^{-1}] \tag{7}$$

$$P_B/\tilde{\mathcal{I}} \simeq \mathcal{R} \otimes_{\|\sqcup\|} \|\sqcup, \sqcup^{-\infty}\| \quad (8)$$

The automorphism

$$P_B \simeq P_B, \quad x_\alpha \mapsto t^{N_\alpha} x_\alpha$$

induces an isomorphism

$$P_B/\tilde{I} \simeq P_B/\tilde{\mathcal{I}} \quad (9)$$

From (??), (??), (??) we obtain (b). Finally, it remains to show (a).

Let $X_\alpha = \bar{x}_\alpha$ (in $\mathcal{R} = \mathcal{P}_A/\mathcal{I}$), $f_\alpha = t^{N_\alpha} X_\alpha$ and

$$\mathcal{M} = \{\{\alpha_\infty \dots \alpha_\nabla \mid \alpha_\infty \geq \dots \geq \alpha_\nabla, \nabla \in \mathbb{Z}_+\}.$$

We shall now show that \mathcal{M} is a $k[t]$ -basis for \mathcal{R} . We first observe that \mathcal{M} is a $k[t]$ -basis for \mathcal{R} . We first observe that any monomial $F = f_{\tau_1} \dots f_{\tau_r}$ is in fact standard. In order to see this, let i be such that $\tau_i \not\geq \tau_{i+1}$. Then using the relation

$$X_{\tau_i} X_{\tau_{i+1}} = X_{\tau_i \vee \tau_{i+1}} X_{\tau_i \wedge \tau_{i+1}} t^{N_{\tau_i \vee \tau_{i+1}} + N_{\tau_i \wedge \tau_{i+1}} - N_{\tau_i} - N_{\tau_{i+1}}},$$

we obtain $F = f_{\tau_1} \dots f_{\tau_{i-1}} f_{\tau_i \vee \tau_{i+1}} f_{\tau_i \wedge \tau_{i+1}} \dots f_{\tau_r}$. Continuing thus, we find (as in the proof of Proposition ??) that at each step the expression for F is lexicographically greater than the expression at the previous step (here we suppose $\tau_1 \succeq \dots \succeq \tau_r$), and thus we arrive at an expression for F as a standard monomial (since \mathcal{L} is finite).

It remains to prove the linear independence of \mathcal{M} . Since standard monomials form a basis for R (cf. Theorem ??), we obtain (by base change), that \mathcal{M} is a $k[t, t^{-1}]$ -basis for $R[t, t^{-1}]$. Denoting the isomorphism $P_B/\tilde{\mathcal{I}} \simeq R[t, t^{-1}]$ by φ , we have $\{\varphi^{-1}(f_{\alpha_1} \dots f_{\alpha_r}) \mid \alpha_1 \geq \dots \geq \alpha_r, r \in \mathbb{Z}_+\}$ is a $k[t, t^{-1}]$ -basis for $\mathcal{R}[\sqcup^{-\infty}]$. For a monomial $\underline{m} = x_{\tau_1} \dots x_{\tau_r}$ (in $R[t, t^{-1}]$), we have $\varphi^{-1}(\underline{m}) = t^{-N_{\underline{m}}} X_{\tau_1} \dots X_{\tau_r}$, where $N_{\underline{m}} = \sum_{i=1}^r N_{\tau_i}$. Hence we obtain $\{f_{\alpha_1} \dots f_{\alpha_r} \mid \alpha_1 \geq \dots \geq \alpha_r, r \in \mathbb{Z}_+\}$ is a $k[t, t^{-1}]$ -basis for $\mathcal{R}[\sqcup^{-\infty}]$ (since $t^{-N_{\underline{m}}}$ is a unit in $k[t, t^{-1}]$). In particular, we obtain that $\{f_{\alpha_1} \dots f_{\alpha_r} \mid \alpha_1 \geq \dots \geq \alpha_r, r \in \mathbb{Z}_+\}$ is linearly independent over $k[t, t^{-1}]$, and hence over $k[t]$. \square

\square

Combining Theorems ?? and ??, we obtain

Theorem 7.4. *The ring $R(\mathcal{L})$ is Cohen-Macaulay.*

8. THE DISTRIBUTIVE LATTICE $I_{d,n}$ AND THE VARIETY $X_{d,n}$

Let

$$I_{d,n} = \{\tau = (i_1, \dots, i_d) \mid 1 \leq i_1 < \dots < i_d \leq n\}.$$

We consider the partial order \geq on $I_{d,n}$ given by

$$(i_1, \dots, i_d) \geq (j_1, \dots, j_d) \iff i_1 \geq j_1, \dots, i_d \geq j_d.$$

For $\tau \in I_{d,n}$, we denote the j -th entry in τ by $\tau(j)$, $1 \leq j \leq d$.

The following result is well-known (see [?] for example):

Proposition 8.1. *$(I_{d,n}, \geq)$ is a distributive lattice.*

For the rest of this section, the lattice $I_{d,n}$ will be denoted by simply \mathcal{L} , and we use the notations introduced in Section ??.

In the discussion below, by a *segment* we shall mean a set consisting of consecutive integers.

Lemma 8.2. *We have*

(a) *The element $\tau = (i_1, \dots, i_d)$ is join-irreducible if and only if either τ is a segment, or τ consists of two disjoint segments (μ, ν) , with μ starting with 1.*

(b) *The element $\tau = (i_1, \dots, i_d)$ is meet-irreducible if and only if either τ is a segment, or τ consists of two disjoint segments (μ, ν) , with ν ending with n .*

(c) *The element $\tau = (i_1, \dots, i_d)$ is join-irreducible and meet-irreducible if and only if either τ is a segment, or τ consists of two disjoint segments (μ, ν) , with μ starting with 1 and ν ending with n .*

Proof. We first observe that (i_1, \dots, i_d) is join-irreducible if and only if $(n+1-i_d, \dots, n+1-i_1)$ is meet-irreducible. Thus it suffice to prove part (a). It is easily seen that $\tau = (i_1, \dots, i_d)$ is a cover for $(j_1, \dots, j_d) \in \mathcal{L}$ if and only if $\{j_1, \dots, j_d\}$ is obtained from (i_1, \dots, i_d) by replacing i_t by $i_t - 1$ for precisely one t , and this proves (a). Part (c) follows from (a) and (b). \square

8.3. For a join-irreducible or meet-irreducible element $\tau \in \mathcal{L}$ we say that τ is of Type I (resp. Type II) if τ consists of just one segment (resp. two disjoint segments), as in the description given by Lemma ??. We denote by $J_{\mathcal{L}}^{(I)}$, $JM_{\mathcal{L}}^{(I)}$ (resp. $J_{\mathcal{L}}^{(II)}$, $JM_{\mathcal{L}}^{(II)}$) respectively the set of elements of $J_{\mathcal{L}}$, $JM_{\mathcal{L}}$ of Type I (resp. Type II). Note that

$$JM_{\mathcal{L}}^{(I)} = \{(i+1, \dots, i+d) \mid 0 \leq i \leq n-d\},$$

and

$$JM_{\mathcal{L}}^{(II)} = \{(1, \dots, j, n+j+1-d, \dots, n) \mid 1 \leq j \leq d-1\}.$$

For $\tau_i = (i+1, \dots, i+d) \in JM_{\mathcal{L}}^{(I)}$ and $\phi_j = ((1, \dots, j, n+j+1-d, \dots, n) \in JM_{\mathcal{L}}^{(II)}$, let $\lambda_{ij} = \tau_i \vee \phi_j$, $\mu_{ij} = \tau_i \wedge \phi_j$. Note that $\tau_0 = \widehat{0}$ and $\tau_{n-d} = \widehat{1}$. Clearly,

$$\lambda_{ij} = (i+1, \dots, i+j, n+1+j-d, \dots, n), \quad \mu_{ij} = (1, \dots, j, i+j+1, \dots, i+d).$$

In the sequel, $X_{d,n}$ will denote the variety $X(\mathcal{L})$, for $\mathcal{L} = \mathcal{I}_{[\setminus]}$.

9. THE IRREDUCIBLE COMPONENTS OF $X_{d,n} \setminus X_{d,n}^\circ$

We preserve the notations of Section ?? . As in Section ?? , let $X_{d,n}^\circ = \{(x_1, \dots, x_N) \in X_{d,n} \mid x_i \neq 0 \text{ for all } i\}$.

Lemma 9.1. *Let $\theta \in J_{\mathcal{L}} \setminus JM_{\mathcal{L}}$, say $\theta = (1, \dots, j, t+1, \dots, t+d-j)$, where $j < t < n+j-d$. Then $\theta = \mu_{ij}$, where $i = t-j$.*

The result follows from the definition of μ_{ij} (note that $0 < i < n-d$).

Let

$$\mathcal{L}_\rangle^{(\nabla)} = \mathcal{L} \setminus \{\tau \in \mathcal{L} \mid \tau(\nabla) = \rangle + \nabla\}$$

where $0 \leq i \leq n-d$, and $1 \leq r \leq d$.

Proposition 9.2. *With notations as above, $\mathcal{L}_\rangle^{(\nabla)}$ is an embedded sublattice of \mathcal{L} .*

Proof. Let $\gamma, \delta \in \mathcal{L}_\rangle^{(\nabla)}$. Then clearly $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_\rangle^{(\nabla)}$. Similarly, if γ, δ in \mathcal{L} are such $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_\rangle^{(\nabla)}$, again it is clear that $\gamma, \delta \in \mathcal{L}_\rangle^{(\nabla)}$. \square

Proposition 9.3. *Let notations be as above. For $0 \leq i \leq n-d$, let*

$$Y_i = X_{d,n} \cap \{x_{\tau_i} = 0\}.$$

Then

- (1) Y_0 and Y_{n-d} are irreducible
- (2) the irreducible components of Y_i , $1 \leq i \leq n-d-1$, are precisely $X(\mathcal{L}_\rangle^{(\nabla)})$, $1 \leq r \leq d$.

Proof. In order to prove (1), it is enough to observe that $Y_0 = X(\mathcal{L} \setminus \{\hat{\uparrow}\})$, $Y_{n-d} = X(\mathcal{L} \setminus \{\widehat{\infty}\})$. Now we prove (2). We have, in view of Propositions ?? and ?? , that $X(\mathcal{L}_\rangle^{(\nabla)})$, $1 \leq r \leq d$ is an irreducible subvariety of $X(\mathcal{L})$. Also $X(\mathcal{L}_\rangle^{(\nabla)}) \subset \mathcal{Y}_\rangle$ (since $\tau_i \notin \mathcal{L}_\rangle^{(\nabla)}$).

Let now $Z \subset Y_i$, Z irreducible. Let η be the generic point of Z . Let $\tau \in \mathcal{L}$, $\tau \neq \tau_i$ such that $\eta_\tau = 0$ (note that, since $\eta_{\tau_i} = 0$, for at least one $\tau \in \mathcal{L}$, $\tau \neq \tau_i$, $\eta_\tau = 0$).

Claim . $\tau \in \mathcal{L}_\rangle^{(\nabla)}$, for some r , $1 \leq r \leq d$.

Proof. Assume that $\tau \notin \mathcal{L}_\rangle^{(\nabla)}$, for all $1 \leq r \leq d$. This implies $\tau(r) = i+r$ for all $1 \leq r \leq d$. But this implies that $\tau = \tau_i$, contradicting the choice of τ . Hence our assumption is wrong, and the Claim follows. \square

Now the Claim implies that $\eta \in X(\mathcal{L}_\rangle^{(\nabla)})$, for some r , $1 \leq r \leq d$, and hence $Z \subset X(\mathcal{L}_\rangle^{(\nabla)})$. \square

Corollary 9.4. *The subvariety $X(\mathcal{L}_j^{(\nabla)})$, $1 \leq i \leq n-d-1$, $1 \leq r \leq d$, of $X_{d,n}$, has codimension 1.*

9.5. We next determine the irreducible components of $X_{d,n} \cap \{x_\phi = 0\}$, for $\phi \in JM_{\mathcal{L}}$ of Type II. Let

$$\phi = \phi_j = (1, \dots, j, n+1+j-d, \dots, n),$$

where $1 \leq j \leq d-1$. Fix r , with $j+1 \leq r \leq n+j-d$, and let

$$\begin{aligned} \lambda_j^{(r)} &= (r-j, r-j+1, \dots, r-1, n+j+1-d, n+j+2-d, \dots, n) \\ \mu_j^{(r)} &= (1, \dots, j, r+1, r+2, \dots, r+d-j), \end{aligned}$$

where $\lambda_j^{(r)}(j) = r-1$. Note that

$$\theta \in [\mu_j^{(r)}, \lambda_j^{(r)}] \iff \theta(j+1) \geq r+1, \text{ and } \theta(j) \leq r-1$$

(and hence $\theta \notin [\mu_j^{(r)}, \lambda_j^{(r)}] \iff \theta(j+1) \leq r$, or $\theta(j) \geq r$). Let

$$\mathcal{L}_j^{(\nabla)} = \mathcal{L} \setminus [\mu_j^{(\nabla)}, \lambda_j^{(\nabla)}],$$

where $1 \leq j \leq d-1$, $j+1 \leq r \leq n+j-d$.

Proposition 9.6. *With notations as above, $\mathcal{L}_j^{(\nabla)}$ is an embedded sublattice of \mathcal{L} .*

Proof. Let $\gamma, \delta \in \mathcal{L}_j^{(\nabla)}$. Then either $\gamma(j) \geq r$, or $\gamma(j+1) \leq r$, and $\delta(j) \geq r$, or $\delta(j+1) \leq r$.

Claim . $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_j^{(\nabla)}$.

Proof. We distinguish four cases.

Case 1: $\gamma(j) \geq r, \delta(j) \geq r$

This implies that $(\gamma \vee \delta)(j) \geq r$, $(\gamma \wedge \delta)(j) \geq r$, and hence $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_j^{(\nabla)}$.

Case 2: $\gamma(j) \geq r, \delta(j+1) \leq r$

This implies that $(\gamma \vee \delta)(j) \geq r$, $(\gamma \wedge \delta)(j+1) \leq r$, and hence $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_j^{(\nabla)}$.

Case 3: $\gamma(j+1) \leq r, \delta(j) \geq r$

This is similar to the previous case.

Case 4: $\gamma(j+1) \leq r, \delta(j+1) \leq r$

In this case, $(\gamma \vee \delta)(j+1) \leq r$, $(\gamma \wedge \delta)(j+1) \leq r$, and hence $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_j^{(\nabla)}$. □

Claim . *Let γ, δ be two noncomparable elements of \mathcal{L} such that $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_j^{(\nabla)}$. Then $\gamma, \delta \in \mathcal{L}_j^{(\nabla)}$.*

Proof. Here, we distinguish three cases.

Case 1: $(\gamma \wedge \delta)(j) \geq r$

This implies $\gamma(j) \geq r$, $\delta(j) \geq r$. Hence $\gamma, \delta \in \mathcal{L}_1^{(\nabla)}$.

Case 2: $(\gamma \vee \delta)(j+1) \leq r$

This implies $\gamma(j+1) \leq r$, $\delta(j+1) \leq r$. Hence $\gamma, \delta \in \mathcal{L}_1^{(\nabla)}$.

Case 3: $(\gamma \vee \delta)(j) \geq r$, $(\gamma \wedge \delta)(j+1) \leq r$

Now $(\gamma \vee \delta)(j) \geq r$ implies that at least one of $\{\gamma(j), \delta(j)\}$ is $\geq r$. Assume that $\gamma(j) \geq r$. This implies $\gamma(j+1) \geq r+1$, and hence $\delta(j+1) \leq r$ (since $(\gamma \wedge \delta)(j+1) \leq r$). Thus, $\gamma(j) \geq r$, and $\delta(j+1) \leq r$. Hence $\gamma, \delta \in \mathcal{L}_1^{(\nabla)}$. \square

The above two claims show that $\mathcal{L}_1^{(\nabla)}$ is an embedded sublattice. \square

Proposition 9.7. *Let notations be as above. For $1 \leq j \leq d-1$, let*

$$Y_j = X_{d,n} \cap \{x_{\phi_j} = 0\}.$$

Then the irreducible components of Y_j are precisely $X(\mathcal{L}_1^{(\nabla)})$, $j+1 \leq r \leq n+j-d$.

Proof. We have, in view of Propositions ?? and ??, that $X(\mathcal{L}_1^{(\nabla)})$ is an irreducible subvariety of $X(\mathcal{L})$. Also $X(\mathcal{L}_1^{(\nabla)}) \subset \mathcal{Y}_1$ (since $\phi_j \notin \mathcal{L}_1^{(\nabla)}$). Let now $Z \subset Y_j$, Z irreducible. Let η be the generic point of Z . Consider an element $\tau \in \mathcal{L}$, $\tau \neq \phi_j$ such that $\eta_\tau = 0$ (note that, since $\eta_{\phi_j} = 0$, for at least one $\tau \in \mathcal{L}$, $\tau \neq \phi_j$, $\eta_\tau = 0$).

Claim . $\tau \in \mathcal{L}_1^{(\nabla)}$, for some $j+1 \leq r \leq n+j-d$.

Proof. Assume that $\tau \notin \mathcal{L}_1^{(\nabla)}$, for all $j+1 \leq r \leq n+j-d$. Then for $r = j+1$, we have $\tau(j) < j+1$. Hence we obtain $\tau(j) = j$. Similarly, for $r = n+j-d$, we have $\tau(j+1) > n+j-d$. Hence $\tau(j+1) = n+1+j-d$ (note that $\tau(j+1) \leq n+1+j-d$ for any $\tau \in \mathcal{L}$). Thus we obtain that $\tau = (1, \dots, j, n+1+j-d, \dots, n)$, i.e. $\tau = \phi_j$, which is not true, by our choice of τ . Hence our assumption is wrong, and the Claim follows \square

Now, Claim implies that $\eta \in X(\mathcal{L}_1^{(\nabla)})$, for some r , with $j+1 \leq r \leq n+j-d$, and hence $Z \subset X(\mathcal{L}_1^{(\nabla)})$, for some r , with $j+1 \leq r \leq n+j-d$. \square

Corollary 9.8. *The subvariety $X(\mathcal{L}_1^{(\nabla)})$, $1 \leq j \leq d-1$, $j+1 \leq r \leq n+j-d$, of $X_{d,n}$, has codimension 1.*

Proposition 9.9. *The irreducible components of $X_{d,n} \setminus X_{d,n}^\circ$ are precisely the subvarieties $X(\mathcal{L}_i^{(\nabla)})$, $1 \leq i \leq n-d-1$, $1 \leq r \leq d$, $X(\mathcal{L}_1^{(\nabla)})$, $1 \leq j \leq d-1$, $j+1 \leq r \leq n+j-d$, and $X(\mathcal{L} \setminus \{\hat{\tau}\})$, $X(\mathcal{L} \setminus \{\infty\})$.*

Proof. Let $Z \subset X_{d,n} \setminus X_{d,n}^\circ$, Z irreducible, and let η be the generic point of Z . In view of Propositions ?? and ??, it is enough to show that there exists a $\tau \in JM_{\mathcal{L}}$ such that $\eta_\tau = 0$.

Assume that $\eta_\tau \neq 0$, for any $\tau \in JM_{\mathcal{L}}$. Let $\theta \in J_{\mathcal{L}} \setminus JM_{\mathcal{L}}$. Then θ is a Type I join-irreducible element of \mathcal{L} . Let

$$\theta = (1, \dots, j, t+1, \dots, t+d-j).$$

We have $\theta = \mu_{ij}$, where $i = t - j$ (cf. Lemma ??). Thus $\theta = \tau_i \wedge \phi_j$, and hence our assumption implies that $\eta_\theta \neq 0$. This, together with our assumption, implies that $\eta_\delta \neq 0$, for any $\delta \in J_{\mathcal{L}}$. This implies $\eta \in X_{d,n}^\circ$ (cf. Lemma ??), which is not true. Hence our assumption is wrong. \square

Lemma 9.10. *We have*

(a) for $1 \leq i \leq n - d - 1$, $JM_{\mathcal{L}} \setminus \{\tau_i\} \subset \mathcal{L}_>^{(\nabla)}$, where $\tau_i = (i + 1, \dots, i + d)$,

(b) for $1 \leq j \leq d - 1$, $JM_{\mathcal{L}} \setminus \{\phi_j\} \subset \mathcal{L}_|^{(\nabla)}$, where $\phi_j = (1, \dots, j, n + j + 1 - d, \dots, i + d)$.

Proof. (a) Let $\theta \in JM_{\mathcal{L}}$. First, let θ be of Type I, say $\theta = (j + 1, \dots, j + d)$, where $j \neq i$. Then, clearly, $\theta \in \mathcal{L}_>^{(\nabla)}$. Now let θ be of Type II, say $\theta = (1, \dots, j, n + 1 + j - d, \dots, n)$. Then $\theta(r) = r$ if $r \leq j$, and $\theta(r) = n - d + r$ if $r > j$. In either case, it is clear that $\theta \in \mathcal{L}_>^{(\nabla)}$.

(b) Let $\theta \in JM_{\mathcal{L}}$. First, let θ be of Type I, say $\theta = (i + 1, i + 2, \dots, i + d)$. We have $\theta(j) = i + j$, $\theta(j + 1) = i + j + 1$. Hence $\theta(j) \geq r$, if $i + j \geq r$, and $\theta(j + 1) \leq r$, if $i + j + 1 \leq r$. Thus $\theta \in \mathcal{L}_|^{(\nabla)}$.

Now let θ be of Type II, say $\theta = (1, \dots, s, n + 1 + s - d, \dots, n)$. If $j < s$, then $\theta(j + 1) = j + 1 \leq r$, and hence $\theta \in \mathcal{L}_|^{(\nabla)}$. If $j \geq s$, then $\theta(j) = n - d + j$, and $\theta(j + 1) = n - d + j + 1$. Hence either $n - d + j + 1 \leq r$, or $n - d + j \geq r$. Thus $\theta \in \mathcal{L}_|^{(\nabla)}$. \square

Proposition 9.11. *Let Y be an irreducible component of $X_{d,n} \setminus X_{d,n}^\circ$. Let P be the generic point of Y . Then $P_\tau = 0$, for precisely one element $\tau \in JM_{\mathcal{L}}$.*

Proof. If Y is equal to $X(\mathcal{L} \setminus \{\hat{\tau}\})$ or $X(\mathcal{L} \setminus \{\widehat{\infty}\})$ (cf. Proposition ??), the assertion is obvious. For the other components the result follows from Proposition ??, and Lemma ??. \square

10. NORMALITY OF $X_{d,n}$

In this section we prove that the variety $X_{d,n}$ is normal. We already know that $X_{d,n}$ is Cohen-Macaulay (cf. Section ??), and in view of Serre

Criterion, it suffices to show that $X_{d,n}$ is nonsingular in codimension 1. In view of Proposition ??, it is enough to prove that generic points of $X(\mathcal{L} \setminus \{\uparrow\})$, $X(\mathcal{L} \setminus \{\infty\})$, $X(\mathcal{L}_j^{(\nabla)})$, $1 \leq i \leq n-d-1$, $1 \leq r \leq d$, $X(\mathcal{L}_j^{(\nabla)})$, $1 \leq j \leq d-1$, $j+1 \leq r \leq n+j-d$ are smooth points. We prove this using the Jacobian Criterion.

Let \mathcal{J} be the Jacobian matrix of $X_{d,n}$, where the rows of \mathcal{J} are indexed by

$$\{f_{\tau,\phi} = x_\tau x_\phi - x_{\tau \vee \phi} x_{\tau \wedge \phi} \mid \tau, \phi \in \mathcal{L} \text{ noncomparable} \},$$

and the columns are indexed by $\{x_\theta \mid \theta \in \mathcal{L}\}$. For simplicity, sometimes we consider the rows being indexed by the diamonds in \mathcal{L} , and the columns being indexed by just the elements of \mathcal{L} .

Recall that for any $P \in X_{d,n}$, if \mathcal{J}_P is the Jacobian matrix evaluated at P , then $\text{rank } \mathcal{J}_P \leq \text{codim}_{\mathbb{A}(\mathcal{L})} \mathcal{X}_{[\cdot, \cdot]}$, with equality if and only if P is a smooth point.

Proposition 10.1. *Let $P \in X_{d,n}$ be such that $P_\tau \neq 0$ for all $\tau \in JM_{\mathcal{L}}^{(I)} \setminus \{\tau_0, \tau_{n-d}\}$. Then P is a smooth point.*

Proof. Let \mathcal{J}_P be the Jacobian matrix evaluated at P . We shall now exhibit a submatrix of \mathcal{J}_P of maximal rank, equal to

$$\text{codim}_{\mathbb{A}(\mathcal{L})} X_{d,n} = \binom{n}{d} - d(n-d) - 1.$$

For each i , $2 \leq i \leq n-d$, let

$$Z_i = \{\theta \in \mathcal{L} \mid \theta([\cdot]) = \rangle + [\cdot] \setminus \{\tau_{|\cdot|}, \cdot \leq |\cdot| \leq [\cdot] - \infty\},$$

where for $0 \leq j \leq d-1$,

$$\tau_{ij} = (i, i+1, \dots, i+j-1, i+j+1, i+j+2, \dots, i+d)$$

(note that $\tau_{i-1} < \tau_{id-1} < \dots < \tau_{i0} = \tau_i$, where $\tau_{i-1} = (i, i+1, \dots, i+d-1)$ and $\tau_i = (i+1, i+2, \dots, i+d)$, as defined in Section ??).

Now, τ_{ij} , $0 \leq j \leq d-1$ are precisely the elements with the properties $\tau_{ij}(d) = i+d$, and $\tau_{ij} > \tau_{i-1}$. Hence τ_{i-1} is not comparable with any θ in Z_i . Now, for each $\theta \in Z_i$, we associate the row indexed by $f_{\tau_{i-1}, \theta}$. We consider the submatrix \mathcal{J}' of \mathcal{J}_P with columns indexed by $\{\theta \in Z_i \mid 2 \leq i \leq n-d\}$, and rows indexed by $\{f_{\tau_{i-1}, \theta} \mid \theta \in Z_i, 2 \leq i \leq n-d\}$. Then \mathcal{J}' is a square matrix, of size equal to $\sum_{i=2}^{n-d} \#Z_i$ (note that the Z_i 's are disjoint). Now the set $\{\theta \in \mathcal{L} \mid \theta([\cdot]) = \rangle + [\cdot]\}$ is in bijection with $I_{d-1, \{1, \dots, i+d-1\}}$ (here, for $r < s$, $I_{r, \{n_1, \dots, n_s\}}$ denotes the set of all r -tuples $\underline{i} = (i_1, \dots, i_r)$, where $i_1 < \dots < i_r$, and $\{i_1, \dots, i_r\} \subset \{n_1, \dots, n_s\}$).

Hence

$$\#Z_i = \binom{i+d-1}{d-1} - d,$$

and

$$\begin{aligned} \# \bigcup_{i=2}^{n-d} Z_i &= \sum_{i=2}^{n-d} \left[\binom{i+d-1}{d-1} - d \right] = \binom{n}{d} - \binom{d}{d-1} - 1 - d(n-d-1) \\ &= \binom{n}{d} - d(n-d) - 1 \end{aligned}$$

(note that $\sum_{j=1}^{m-r+1} \binom{m-j}{r-1} = \binom{m}{r}$).

Now, a typical row in \mathcal{J}' is indexed by

$$f_{\tau_{i-1}, \theta} = x_{\tau_{i-1}} x_{\theta} - x_{\tau_{i-1} \vee \theta} x_{\tau_{i-1} \wedge \theta},$$

where $\theta \in Z_i$. The only variable appearing in $f_{\tau_{i-1}, \theta}$, which is an index for a column of \mathcal{J}' , is x_{θ} . Further, the entry in \mathcal{J}' in the row indexed by $f_{\tau_{i-1}, \theta}$ and column indexed by x_{θ} is nonzero (by hypothesis). Thus, \mathcal{J}' is a diagonal matrix, with nonzero diagonal entries. Hence \mathcal{J}' has rank equal to its size, which is

$$\binom{n}{d} - d(n-d) - 1 = \text{codim}_{\mathbb{A}(\mathcal{L})} X_{d,n}.$$

Thus we obtain that P is a smooth point of $X_{d,n}$. □

Proposition 10.2. *Let $P \in X_{d,n}$ be such that $P_{\tau} \neq 0$, for all $\tau \in JM_{\mathcal{L}}^{(II)}$. Then P is a smooth point of $X_{d,n}$.*

Proof. Let $\mathcal{J}_{\mathcal{P}}$ be the Jacobian matrix evaluated at P . We shall now exhibit a submatrix of $\mathcal{J}_{\mathcal{P}}$ of maximal rank. Fix r and i , $1 \leq r \leq d-1$, $r+1 \leq i \leq n-d+r-1$. Set

$$Z_r^{(i)} = \{\theta \in \mathcal{L} \mid \theta(\nabla) = \rangle, \text{ and } \theta(\sqcup) = \sqcup, \text{ for } \sqcup < \nabla\} \setminus \{\theta\},$$

where $\theta_i = (1, \dots, r-1, i, n+r-d+1, \dots, n)$. Let

$$Z_r = \bigcup_{i=r+1}^{n-d+r-1} Z_r^{(i)}.$$

Then

$$\bigcup_{s \leq r} Z_s = \{\tau \in \mathcal{L} \mid \tau \text{ and } \phi_{\nabla} \text{ are noncomparable}\},$$

where $\phi_r = (1, \dots, r, n+r-d+1, \dots, n)$, $1 \leq r \leq d-1$, as defined in Section ???. Let \mathcal{J}' be the submatrix of $\mathcal{J}_{\mathcal{P}}$ with columns indexed by $\{\theta \in Z_r \mid 1 \leq r \leq d-1\}$, and rows indexed by $\{f_{\phi_r, \theta} \mid \theta \in Z_r, 1 \leq r \leq d-1\}$. Then \mathcal{J}' is a square matrix of size equal to $\sum_{r=1}^{d-1} \#Z_r$

(note that the Z_r 's are disjoint). Now, $\{\theta \in \mathcal{L} \mid \theta(\nabla) = \square\}$, and $\theta(\square) = \square$, for $\square < \nabla\}$ is in bijection with $I_{d-r, \{i+1, \dots, n\}}$. Hence

$$\#Z_r^{(i)} = \binom{n-i}{d-r} - 1,$$

and

$$\begin{aligned} \#Z_r &= \sum_{i=r+1}^{n-d+r-1} \left[\binom{n-i}{d-r} - 1 \right] = \binom{n-r}{d-r+1} - 1 - (n-d-1) \\ &= \binom{n-r}{d-r+1} - (n-d) = \binom{n-r}{n-d-1} - (n-d). \end{aligned}$$

Hence the size of \mathcal{J}' is equal to

$$\begin{aligned} \sum_{r=1}^{d-1} \left[\binom{n-r}{n-d-1} - (n-d) \right] &= \binom{n}{n-d} - \binom{n-d}{n-d-1} - 1 - (d-1)(n-d) \\ &= \binom{n}{d} - (n-d) - 1 - (d-1)(n-d) = \binom{n}{d} - d(n-d) - 1. \end{aligned}$$

As in the proof of Proposition ??, we find that \mathcal{J}' is a diagonal matrix, with nonzero diagonal entries. Hence the rank of \mathcal{J}' is equal to its size, which is

$$\binom{n}{d} - d(n-d) - 1 = \text{codim}_{\mathbb{A}(\mathcal{L})} X_{d,n}.$$

Thus P is a smooth point of $X_{d,n}$. \square

Proposition 10.3. *The variety $X_{d,n}$ is nonsingular in codimension 1.*

Proof. Let Y be an irreducible component of $X_{d,n} \setminus X_{d,n}^\circ$. Let P be the generic point of Y . Then, by Proposition ??, $P_\tau = 0$ for precisely one element $\tau \in JM_{\mathcal{L}}$. Hence the result follows from Propositions ?? and ??. \square

Proposition ?? and Theorem ?? yield the following

Theorem 10.4. *The variety $X_{d,n}$ is a normal toric variety.*

10.5. Let $A_{d,n}$ denote the homogeneous coordinate ring of $G_{d,n}$ (the Grassmannian of d -planes in k^n) for the Plücker embedding. We recall (cf. [?]):

Theorem 10.6. *There exists a flat family whose general fiber is $A_{d,n}$, and whose special fiber is $R(I_{d,n})$.*

Remark 10.7. The above result is also proved in [?] using SAGBI theory.

Combining Theorems ?? and ??, we obtain

Theorem 10.8. *The Grassmannian $G_{d,n}$ degenerates to the normal toric variety $X_{d,n}$.*

11. A CONJECTURE ON THE SINGULAR LOCUS OF $X_{d,n}$

In this section we prove a partial result towards the determination of the singular locus of $X_{d,n}$. We also state a conjecture regarding the description of the singular locus of $X_{d,n}$.

We assume that $n \geq 4$ (note that for $n \leq 3$, $X_{d,n}$ is smooth). Let notations be as in Section ???. Let $\mathcal{L}_{\lceil j} = \mathcal{L} \setminus [\mu_{\lceil j}, \lambda_{\lceil j}]$, $0 \leq i \leq n - d$, $1 \leq j \leq d - 1$.

Lemma 11.1. *Let $i = 0$, and $1 \leq j \leq d - 1$. Then $\mathcal{L}_{\lceil j}$ is an embedded sublattice. Further, $JM^{(I)} \setminus \{\tau_0\} \subset \mathcal{L}_{\lceil j}$.*

Proof. We have $\tau_0 = (1, \dots, d)$, $\phi_j = (1, \dots, j, n + 1 + j - d, \dots, n)$, and hence $\mu_{0j} = \tau_0$, and $\lambda_{ij} = \phi_j$. Hence $\theta \in [\tau_0, \phi_j]$ if and only if $\theta(t) = t$, for $1 \leq t \leq j$. Therefore $\theta \in \mathcal{L}_{\lceil j}$ if and only if $\theta(t) \neq t$, for some t , $1 \leq t \leq j$. Let γ, δ be two noncomparable elements in $\mathcal{L}_{\lceil j}$. Let $t_1, t_2 \leq j$ be such that $\gamma(t_1) \neq t_1$, $\delta(t_2) \neq t_2$. Then, letting $t = \max(t_1, t_2)$, we have $\gamma \vee \delta(t) \neq t$, $\gamma \wedge \delta(t) \neq t$. Thus $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_{\lceil j}$.

Let now γ, δ be two noncomparable elements of \mathcal{L} such that $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_{\lceil j}$. Let $t \leq j$ be such that $\gamma \wedge \delta(t) \neq t$. This implies $\gamma \vee \delta(t) \neq t$. Then, clearly $\gamma(t) \neq t$, $\delta(t) \neq t$. Thus $\gamma, \delta \in \mathcal{L}_{\lceil j}$. Thus we obtain that $\mathcal{L}_{\lceil j}$ is an embedded sublattice. Let $\tau \in JM^{(I)} \setminus \{\tau_0\}$, say $\tau = (i + 1, \dots, i + d)$, where $i \geq 1$. Then clearly $\tau \in \mathcal{L}_{\lceil j}$ (note for example that $\tau(1) \neq 1$). \square

Corollary 11.2. *Let P be the generic point of $X(\mathcal{L}_{\lceil j})$. Then P is a smooth point of \mathcal{L} .*

Proof. We have $P_\tau \neq 0$, for $\tau \in JM^{(I)} \setminus \{\tau_0\}$. Hence the result follows from Proposition 10.1. \square

Lemma 11.3. *Let $i = n - d$, and $1 \leq j \leq d - 1$. Then $\mathcal{L}_{\setminus \lceil j}$ is an embedded sublattice. Further, $JM^{(I)} \setminus \{\tau_{n-d}\} \subset \mathcal{L}_{\setminus \lceil j}$.*

Corollary 11.4. *Let P be the generic point of $X(\mathcal{L}_{\setminus \lceil j})$. Then P is a smooth point of \mathcal{L} .*

The proofs of Lemma 11.3, and Corollary 11.4 are similar to those of Lemma 11.1, and Corollary 11.2 respectively.

Lemma 11.5. *Let $1 \leq i \leq n - d - 1$, $1 \leq j \leq d - 1$. Then $\mathcal{L}_{\lceil j}$ is an embedded sublattice, and*

$$\text{codim}_{\mathbb{A}(\mathcal{L}_{\lceil j})} X(\mathcal{L}_{\lceil j}) < \text{codim}_{\mathbb{A}(\mathcal{L})} \mathcal{X}(\mathcal{L}).$$

Proof. For $\theta \in \mathcal{L}$ we have

$$\theta \in [\mu_{ij}, \lambda_{ij}] \iff \theta(j) \leq i + j, \text{ and } \theta(j + 1) \geq i + j + 1$$

(here, by $\theta(j)$, we mean the j -th entry in the d -tuple θ). Let γ, δ two noncomparable elements in $\mathcal{L}_{\triangleright|}$. We have either $\gamma(j) \geq i + j + 1$, or $\gamma(j + 1) \leq i + j$. Similarly, we have either $\delta(j) \geq i + j + 1$, or $\delta(j + 1) \leq i + j$.

Claim . $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_{\triangleright|}$.

The proof is the same as in the proof of the first Claim in Proposition ??.

Claim . Let γ, δ be two noncomparable elements of \mathcal{L} such that $\gamma \vee \delta, \gamma \wedge \delta \in \mathcal{L}_{\triangleright|}$. Then $\gamma, \delta \in \mathcal{L}_{\triangleright|}$.

The proof is the same as in the proof of the second Claim in Proposition ??.

The above two claims show that $\mathcal{L}_{\triangleright|}$ is an embedded sublattice. It remains to prove the inequality

$$\text{codim}_{\mathbb{A}(\mathcal{L}_{\triangleright|})} X(\mathcal{L}_{\triangleright|}) < \text{codim}_{\mathbb{A}(\mathcal{L})} \mathcal{X}(\mathcal{L}).$$

If $\theta = (k + 1, \dots, k + d) \in J_{\mathcal{L}}^{(I)}$, $k \neq i$, is a join-irreducible element of Type I distinct from τ_i , then $\theta(j) = k + j \geq i + j + 1$, if $k \geq i + 1$, and $\theta(j + 1) = k + j + 1 \leq i + j$, if $k \leq i - 1$. Further, the unique element θ' of \mathcal{L} such that (θ, θ') is a cover in \mathcal{L} is given by $\theta' = (k, k + 2, \dots, k + d)$. It is easily seen that if either $k \neq i + 1$, or $j \neq 1$ then $\theta' \in \mathcal{L}_{\triangleright|}$, and is the unique element of $\mathcal{L}_{\triangleright|}$ such that (θ, θ') is a cover in $\mathcal{L}_{\triangleright|}$. Let then $k = i + 1$, and $j = 1$. In this case, we observe that if $\delta = (i, i + 1, i + 4, \dots, i + d + 1)$, then $\delta \in \mathcal{L}_{\triangleright|}$, and δ is the unique element of $\mathcal{L}_{\triangleright|}$ such that (θ, δ) is a cover in $\mathcal{L}_{\triangleright|}$ (note that $[\delta, \theta] \cap \mathcal{L}_{\triangleright|} = \{\delta, \theta\}$). Thus $\theta \in J_{\mathcal{L}_{\triangleright|}}$.

Let now $\theta = (1, \dots, s, t + 1, \dots, t + d - s) \in J_{\mathcal{L}}^{(II)}$, where $t \neq n + j - d$, if $s = j$, be a join-irreducible element of Type II distinct from ϕ_j . We distinguish three cases.

Case 1: $s < j$

We have $\theta(j) = t + j - s$, $\theta(j + 1) = t + j + 1 - s$. Hence, if $t - s \geq i + 1$, then $\theta(j) \geq i + j + 1$, and $\theta \in \mathcal{L}_{\triangleright|}$. If $t - s \leq i - 1$, then $\theta(j + 1) \leq i + j$, and $\theta \in \mathcal{L}_{\triangleright|}$. Further, the unique element θ' of \mathcal{L} such that (θ, θ') is a cover in \mathcal{L} is given by $\theta' = (1, \dots, s, t, t + 2, \dots, t + d)$. It is easily seen that $\theta' \in \mathcal{L}_{\triangleright|}$, and is the unique element of $\mathcal{L}_{\triangleright|}$ such that (θ, θ') is a cover, except when $s + 1 = j$, and $t - s = i + 1$. Thus $\theta \in J_{\mathcal{L}_{\triangleright|}}$, except when $s + 1 = j$, and $t - s = i + 1$. In this case, we observe that if

$\delta = (1, \dots, s, i + s, i + s + 1, t + 3, \dots, t + d - j)$, then $\delta \in \mathcal{L}_{\triangleright|}$, and is the unique element of $\mathcal{L}_{\triangleright|}$ such that (θ, δ) is a cover in $\mathcal{L}_{\triangleright|}$. Thus $\theta \in J_{\mathcal{L}_{\triangleright|}}$.

Case 2: $s = j$

We have $\theta(j) = j$, $\theta(j + 1) = t + 1$. Hence, if $\theta(j + 1) \leq i + j$, i.e. $t \leq i + j - 1$, then $\theta \in \mathcal{L}_{\triangleright|}$. Further, the unique element θ' of \mathcal{L} such that (θ, θ') is a cover in \mathcal{L} (namely $\theta' = (1, \dots, j, t, t + 2, \dots, t + d - j)$) also belongs to $\mathcal{L}_{\triangleright|}$, and is the unique element of $\mathcal{L}_{\triangleright|}$ such that (θ, θ') is a cover in $\mathcal{L}_{\triangleright|}$. Hence $\theta \in J_{\mathcal{L}_{\triangleright|}}$.

Case 3: $s > j$

We have $s \geq j + 1$, and hence $\theta(j + 1) = j + 1 \leq i + j$. Hence $\theta \in \mathcal{L}_{\triangleright|}$. As in

Case 2, if θ' is the unique element of \mathcal{L} such that (θ, θ') is a cover in \mathcal{L} , then $\theta' \in \mathcal{L}_{\triangleright|}$, and is the unique element of $\mathcal{L}_{\triangleright|}$ such that (θ, θ') is a cover in $\mathcal{L}_{\triangleright|}$. Hence $\theta \in J_{\mathcal{L}_{\triangleright|}}$.

From above, we have

$$\#J_{\mathcal{L}_{\triangleright|}} \geq \#J_{\mathcal{L}} - (n - d - i + j + 1).$$

On the other hand, we have

$$\#J_{\mathcal{L}} - \#J_{\mathcal{L}_{\triangleright|}} = \#[\mu_{ij}, \lambda_{ij}] = \binom{i + j}{j} \binom{n - i - j}{d - j}.$$

It can be seen easily (by assuming $d \leq [n/2]$, since $I_{d,n}$ is isomorphic to $I_{n-d,n}$) that

$$\binom{i + j}{j} \binom{n - i - j}{d - j} > n - d - i + j + 1,$$

i.e.

$$\#\mathcal{L} - \#\mathcal{J}_{\mathcal{L}} > \#\mathcal{L}_{\triangleright|} - \#\mathcal{J}_{\mathcal{L}_{\triangleright|}}.$$

The result now follows from this. □

Proposition 11.6. *For $1 \leq i \leq n - d - 1$, $1 \leq j \leq d - 1$, we have*

$$X(\mathcal{L}_{\triangleright|}) \subset \text{Sing}(\mathcal{X}_{[\cdot, \cdot]}).$$

Proof. Let P be the generic point of $\mathcal{L}_{\triangleright|}$. We have $P_{\tau} = 0$ for all $\tau \in [\mu_{ij}, \lambda_{ij}]$. Let $\mathcal{J}_{\mathcal{P}}$ be the Jacobian matrix evaluated at P . Then the submatrix of $\mathcal{J}_{\mathcal{P}}$ with rows given by $\{f_{\tau, \phi} \mid \tau, \phi \in [\mu_{ij}, \lambda_{ij}], \tau, \phi \text{ noncomparable}\}$, and columns given by $\{x_{\tau} \mid \tau \in [\mu_{ij}, \lambda_{ij}]\}$ is the zero matrix. Let \mathcal{J}' be the matrix obtained from $\mathcal{J}_{\mathcal{P}}$ by deleting these rows and columns. Then we have

$$\text{rank } \mathcal{J}' = \text{rank } \mathcal{J}_{\mathcal{P}}. \tag{1}$$

Further, \mathcal{J}' is precisely the Jacobian matrix of the variety $X(\mathcal{L}_{\gamma|}) \subset \mathbb{A}_{\mathcal{L}_{\gamma|}}$, evaluated at $P' = (P_{\tau})_{\tau \in \mathcal{L}_{\gamma|}}$. Hence

$$\text{rank } \mathcal{J}' \leq \text{codim}_{\mathbb{A}(\mathcal{L}_{\gamma|})} \mathcal{X}(\mathcal{L}_{\gamma|}) < \text{codim}_{\mathbb{A}(\mathcal{L})} \mathcal{X}_{\Gamma, \lambda} \quad (2)$$

(cf. Lemma ??). The result now follows from (1) and (2). \square

We have the following conjecture on the singular locus of $X_{d,n}$.

Conjecture . *The irreducible components of $\text{Sing } X_{d,n}$ are precisely $X(\mathcal{L}_{\gamma|})$, $1 \leq i \leq n - d - 1$, $1 \leq j \leq d - 1$.*

12. GENERALITIES ON G/Q

Let G be a semisimple and simply connected algebraic group, $T \subset G$ a maximal torus, and $B \supset T$, a Borel subgroup. Let R be the root system of G relative to T . Let R^+ (resp. S) be the system of positive (resp. simple) roots of R with respect to B . Let Q be a parabolic subgroup of G containing B . Associated to Q , there is a subset S_Q of S such that Q is the subgroup of G generated by B and $\{U_{-\alpha} \mid \alpha \in R_Q^+\}$, where $R_Q^+ = \{\alpha \in R^+ \mid \alpha = \sum_{\beta \in S_Q} a_{\beta} \beta\}$ (here, for $\beta \in R$, U_{β} denotes the 1 dimensional unipotent subgroup of G). Let W_Q be the Weyl group of Q (note that W_Q is simply the subgroup of W generated by $\{s_{\alpha} \mid \alpha \in S_Q\}$).

12.1. In each coset wW_Q , there exists a unique element of minimal length (cf. [?]). Let W_Q^{\min} be this set of representatives of W/W_Q . The set W_Q^{\min} is called the *set of minimal representatives of W/W_Q* . We have

$$W_Q^{\min} = \{w \in W \mid l(ww') = l(w) + l(w'), \text{ for all } w' \in Q\}.$$

The set W_Q^{\min} may be also be characterized as

$$W_Q^{\min} = \{w \in W \mid w(\alpha) > 0, \text{ for all } \alpha \in S_Q\}$$

(here by a root being > 0 we mean $\beta \in R^+$).

12.2. In each coset wW_Q there exists a unique element of maximal length. Let W_Q^{\max} be the set of these representatives of W/W_Q . We have

$$W_Q^{\max} = \{w \in W \mid w(\alpha) < 0 \text{ for all } \alpha \in S_Q\}.$$

Further, if we denote by w_Q the element of maximal length in W_Q , then we have

$$W_Q^{\max} = \{ww_Q \mid w \in W_Q^{\min}\}.$$

In the sequel, given $w \in W$, the minimal (resp. maximal) representative of wW_Q in W will be denoted by w_Q^{\min} (resp. w_Q^{\max}).

12.3. Maximal parabolic subgroups. The set of maximal parabolic subgroups is in one-to-one correspondence with S , namely given $\alpha \in S$, the parabolic subgroup Q where $S_Q = S \setminus \{\alpha\}$ is a maximal parabolic subgroup, and conversely. We shall denote Q , where $S_Q = S \setminus \{\alpha\}$ by $P_{\hat{\alpha}}$, and refer to it as the *maximal parabolic subgroup obtained by omitting α* .

12.4. Schubert varieties in G/Q . For $w \in W$, let us denote the coset wQ by $e_{w,Q}$. Then the set of T -fixed points in G/Q for the action given by left multiplication is precisely $\{e_{w,Q} \mid w \in W\}$. Let $w \in W$, and let $X_Q(w)$ be the Zariski closure of $Be_{w,Q}Q \pmod{Q}$ in G/Q . Then $X_Q(w)$ with the canonical reduced structure is called the Schubert variety in G/Q associated to w . We have the well-known Bruhat decomposition

$$G/Q = \dot{\cup} Be_{w,Q}, \quad X_Q(\theta) = \dot{\cup}_{w \leq \theta} Be_{w,Q}, \quad \theta \in W.$$

Let us denote by w_Q^{\min} (resp. w_Q^{\max}) the minimal (resp. maximal) representative of wW_Q . Let $\pi : G/B \rightarrow G/Q$ be the canonical projection. Then it can be easily seen that $\pi|_{X_B(w_Q^{\max})} : X_B(w_Q^{\max}) \rightarrow X_Q(w)$ is a fibration with fiber $\simeq Q/B$ while $\pi|_{X_B(w_Q^{\min})} : X_B(w_Q^{\min}) \rightarrow X_Q(w)$ is birational. In particular, we have $\dim X_Q(w) = \dim X_B(w_Q^{\min})$.

12.5. The big cell and the opposite big cell. The B -orbit Be_{w_0} in G/Q (w_0 being the element of maximal length in W) is called the *big cell* in G/Q . It is a dense open subset of G/Q , and it gets identified with $R_u(Q)$, the unipotent radical of Q , namely the subgroup of B generated by $\{U_\alpha \mid \alpha \in R^+ \setminus R_Q^+\}$ (cf. [?]). Let B^- be the Borel subgroup of G opposite to B , i.e. the subgroup of G generated by T and $\{U_{-\alpha} \mid \alpha \in R^+\}$. The B^- -orbit $B^-e_{\text{id},Q}$ is called the *opposite big cell* in G/Q . This is again a dense open subset of G/Q , and it gets identified with the unipotent subgroup of B^- generated by $\{U_{-\alpha} \mid \alpha \in R^- \setminus R_Q^-\}$. Observe that both the big cell and the opposite big cell can be identified with \mathbb{A}^{N_Q} , where $N_Q = \#\{R^+ \setminus R_Q^+\}$.

For a Schubert variety $X(w) \subset G/Q$, $B^-e_{\text{id}} \cap X(w)$ is called the *opposite cell* in $X(w)$ (by abuse of language). In general, it is not a cell (except for $w = w_0$). It is a nonempty affine open subvariety of $X(w)$, and a closed subvariety of the affine space B^-e_{id} .

12.6. Equations defining a Schubert variety. Let L be an ample line bundle on G/Q . Consider the projective embedding $G/Q \hookrightarrow \text{Proj}(H^0(G/Q, L))$. We recall (cf. [?]) that the homogeneous ideal of G/Q for this embedding is generated in degree 2, and any Schubert

variety X in G/Q is scheme theoretically (even at the cone level) the intersection of G/Q with all the hyperplanes in $\text{Proj}(H^0(G/Q, L))$ containing X .

For a maximal parabolic subgroup P_i , let us denote the ample generator of $\text{Pic}(G/P_i) (\simeq \mathbb{Z})$ by L_i .

Given a parabolic subgroup Q , let us denote $S \setminus S_Q$ by $\{\alpha_1, \dots, \alpha_t\}$, for some t . Let

$$R = \bigoplus_{\underline{a}} H^0(G/Q, \bigotimes_i L_i^{a_i})$$

$$R_w = \bigoplus_{\underline{a}} H^0(X_Q(w), \bigotimes_i L_i^{a_i}),$$

where $\underline{a} = (a_1, \dots, a_t) \in \mathbb{Z}_+^t$. Let us denote $\bigotimes_i L_i^{a_i}$ by just L . We recall (cf. [?]) that the natural map

$$\bigoplus \mathcal{S}^{-\infty}(\mathcal{H}'(\mathcal{G}/\mathcal{Q}, \mathcal{L}_\infty)) \otimes \dots \otimes \mathcal{S}^{-\infty}(\mathcal{H}'(\mathcal{G}/\mathcal{Q}, \mathcal{L}_\square)) \rightarrow \mathcal{R}$$

is surjective, and its kernel is generated as an ideal by elements of total degree 2. Further, the restriction map $R \rightarrow R_w$ is surjective, and its kernel is generated as an ideal by elements of total degree 1.

13. OPPOSITE CELLS IN SCHUBERT VARIETIES IN $SL(n)/B$

Let $G = SL(n)$, the special linear group of rank $n - 1$. Let T be the maximal torus consisting of all the diagonal matrices in G , and B the Borel subgroup consisting of all the upper triangular matrices in G . It is well-known that W can be identified with \mathcal{S}_\setminus , the symmetric group on n letters.

Following [?], we denote the simple roots by $\varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq n - 1$ (note that $\varepsilon_i - \varepsilon_{i+1}$ is the character sending $\text{diag}(t_1, \dots, t_n)$ to $t_i t_{i+1}^{-1}$). Then $R = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n\}$, and the reflection $s_{\varepsilon_i - \varepsilon_{i+1}}$ may be identified with the transposition $(i, i+1)$ in \mathcal{S}_\setminus . For $w = (a_1 \dots a_n) \in \mathcal{S}_\setminus$, it is easily seen that $w(\varepsilon_i - \varepsilon_j) = \varepsilon_{a_i} - \varepsilon_{a_j}$.

13.1. The Chevalley-Bruhat order on \mathcal{S}_\setminus . For $w_1, w_2 \in W$, we have

$$X(w_1) \subset X(w_2) \iff \pi_d(X(w_1)) \subset \pi_d(X(w_2)) \text{ for all } 1 \leq d \leq n - 1,$$

where π_d is the canonical projection $G/B \rightarrow G/P_{\hat{\alpha}_d}$. Hence we obtain that for $(a_1 \dots a_n), (b_1 \dots b_n) \in \mathcal{S}_\setminus$,

$$(a_1 \dots a_n) \geq (b_1 \dots b_n) \iff (a_1 \dots a_d) \uparrow \geq (b_1 \dots b_d) \uparrow \text{ for all } 1 \leq d \leq n - 1$$

(here, for an ordered d -tuple $(t_1 \dots t_d)$ of distinct integers, $(t_1 \dots t_d) \uparrow$ denotes the ordered d -tuple obtained from $\{t_1, \dots, t_d\}$ by arranging its elements in ascending order).

13.2. The partially ordered set I_{a_1, \dots, a_k} . Let Q be a parabolic subgroup in $SL(n)$. Let $1 \leq a_1 < \dots < a_k \leq n$, such that $S_Q = S \setminus \{\alpha_{a_1}, \dots, \alpha_{a_k}\}$ (we follow [?] for indexing the simple roots). Then $Q = P_{\alpha_{a_1}} \widehat{\cap} \dots \widehat{\cap} P_{\alpha_{a_k}}$, and $W_Q = \mathcal{S}_{+\infty} \times \mathcal{S}_{+\infty - +\infty} \times \dots \times \mathcal{S}_{\setminus -+}$. Let

$$I_{a_1, \dots, a_k} = \{(\underline{i}_1, \dots, \underline{i}_k) \in I_{a_1, n} \times \dots \times I_{a_k, n} \mid \underline{i}_t \subset \underline{i}_{t+1} \text{ for all } 1 \leq t \leq k-1\}.$$

Then it is easily seen that W_Q^{\min} may be identified with I_{a_1, \dots, a_k} .

The partial order on the set of Schubert varieties in G/Q (given by inclusion) induces a partial order \geq on I_{a_1, \dots, a_k} , namely, for $\mathbf{i} = (\mathbf{i}_1, \dots, \mathbf{i}_k)$, $\mathbf{j} = (\mathbf{j}_1, \dots, \mathbf{j}_k) \in \mathbf{I}_{a_1, \dots, a_k}$, $\mathbf{i} \geq \mathbf{j} \iff \mathbf{i}_t \geq \mathbf{j}_t$ for all $1 \leq t \leq k$.

13.3. The minimal (maximal) representatives as permutations.

Let $w \in W_Q$, and let $\mathbf{i} = (\mathbf{i}_1, \dots, \mathbf{i}_k)$ be the element in I_{a_1, \dots, a_k} which corresponds to w_Q^{\min} . As a permutation, the element w_Q^{\min} is given by \underline{i}_1 , followed by $\underline{i}_2 \setminus \underline{i}_1$ arranged in ascending order, and so on, ending with $\{1, \dots, n\} \setminus \underline{i}_k$ arranged in ascending order. Similarly, as a permutation, the element w_Q^{\max} is given by \underline{i}_1 arranged in descending order, followed by $\underline{i}_2 \setminus \underline{i}_1$ arranged in descending order, etc..

13.4. The opposite big cell in G/Q . Let $Q = \cap_{t=1}^k P_{\alpha_{a_t}}$. Let $a = n - a_k$, and Q be the parabolic subgroup consisting of all the elements of G of the form

$$\begin{pmatrix} A_1 & * & * & \cdots & * & * \\ 0 & A_2 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_k & * \\ 0 & 0 & 0 & \cdots & 0 & A \end{pmatrix},$$

where A_t is a matrix of size $a_t \times a_t$, $1 \leq t \leq k$, and A is a matrix of size $a \times a$, and $x_{ml} = 0$, $m > a_t$, $l \leq a_t$, $1 \leq t \leq k$. Denote by O^- the subgroup of G generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$. Then O^- consists of the elements of G of the form

$$\begin{pmatrix} I_1 & 0 & 0 & \cdots & 0 & 0 \\ * & I_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \cdots & I_k & 0 \\ * & * & * & \cdots & * & I_a \end{pmatrix},$$

where I_t (resp. I_a) is the $a_t \times a_t$ (resp. $a \times a$) identity matrix, and if $x_{ml} \neq 0$, with $m \neq l$, then $m > a_t$, $l \leq a_t$. Further, the restriction of the canonical morphism $f : G \rightarrow G/Q$ to O^- is an open immersion, and $f(O^-) \simeq B^- e_{\text{id}, Q}$. Thus $B^- e_{\text{id}, Q}$ gets identified with O^- .

13.5. Plücker coordinates of the Grassmannian. Let $G_{d,n}$ be the Grassmannian variety, consisting of d -dimensional subspaces of an n -dimensional vector space V . Let us identify V with k^n , and denote the standard basis of k^n by $\{e_i \mid 1 \leq i \leq n\}$. Consider the Plücker embedding $f_d : G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d V)$, where $\wedge^d V$ is the d -th exterior power of V . For $\underline{i} = (i_1, \dots, i_d) \in I_{d,n}$, let $e_{\underline{i}} = e_{i_1} \wedge \dots \wedge e_{i_d}$. Then the set $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ is a basis for $\wedge^d V$. Let us denote the basis of $(\wedge^d V)^*$ (the linear dual of $\wedge^d V$) dual to $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ by $\{p_{\underline{j}} \mid \underline{j} \in I_{d,n}\}$. Then $\{p_{\underline{j}} \mid \underline{j} \in I_{d,n}\}$ gives a system of coordinates for $\mathbb{P}(\wedge^d V)$. These are the so-called *Plücker coordinates*.

13.6. Schubert varieties in the Grassmannian. Let $Q = P_{\hat{\alpha}_d}$. For simplicity of notation, let us denote $P_{\hat{\alpha}_d}$ by just P_d . We have

$$G_{d,n} \simeq G/P_d.$$

Let $\underline{i} = (i_1, \dots, i_d) \in I_{d,n}$. Then the T -fixed point $e_{\underline{i}, P_d}$ is simply the d -dimensional span of $\{e_{i_1}, \dots, e_{i_d}\}$. Thus $X_{P_d}(\underline{i})$ is simply the Zariski closure of $B[e_{i_1} \wedge \dots \wedge e_{i_d}]$ in $\mathbb{P}(\wedge^d V)$.

In view of the Bruhat decomposition for $X_{P_d}(\underline{i})$ (cf. §??), we have

$$p_{\underline{j}}|_{X_{P_d}(\underline{i})} \neq 0 \iff \underline{i} \geq \underline{j}.$$

Now using §??, we obtain

Remark 13.7. Given $\tau \in I_{d,n}$, $\tau = (i_1, \dots, i_d)$, let Λ_τ denote the Young diagram $(i_d - d, \dots, i_1 - 1)$. Then, using Lemma ??, we see that the element $\tau \in I_{d,n}$ is join-irreducible if and only if the associated Young diagram Λ_τ is a rectangle, i.e. all rows have the same number of boxes. We also observe that τ is join-irreducible if and only if $X(\tau)$ is nonsingular (this is a consequence of the fact that $X(\tau)$ is nonsingular if and only if Λ_τ is a rectangle, cf. [?]).

Proposition 13.8. *The restriction map $R \rightarrow R_{\underline{i}}$ is surjective, and the kernel is generated as an ideal by $\{p_{\underline{j}} \mid \underline{i} \not\geq \underline{j}\}$*

13.9. Evaluation of Plücker coordinates on the opposite big cell in G/P_d . Consider the morphism $\phi_d : G \rightarrow \mathbb{P}(\wedge^d V)$, where $\phi_d = f_d \circ \theta_d$, θ_d being the natural projection $G \rightarrow G/P_d$. Then $p_{\underline{j}}(\phi_d(g))$ is simply the minor of g consisting of the first d columns and the rows with indices j_1, \dots, j_d . Now, denote by Z_d the unipotent subgroup of G generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_{P_d}^-\}$. We have, as in §??

$$Z_d = \left\{ \begin{pmatrix} I_{d \times d} & 0_{d \times (n-d)} \\ A_{(n-d) \times d} & I_{(n-d) \times (n-d)} \end{pmatrix} \in G \right\}$$

As in §??, we identify Z_d with the opposite big cell in G/P_d . Then, given $z \in Z_d$, the Plücker coordinate $p_{\underline{j}}$ evaluated at z is simply a certain minor of A , which may be explicitly described as follows. Let $\underline{j} = (j_1, \dots, j_d)$, and let j_r be the largest entry $\leq d$. Let $\{k_1, \dots, k_{d-r}\}$ be the complement of $\{j_1, \dots, j_r\}$ in $\{1, \dots, d\}$. Then this minor of A is given by column indices k_1, \dots, k_{d-r} , and row indices j_{r+1}, \dots, j_d (here the rows of A are indexed as $d+1, \dots, n$). Conversely, given a minor of A , say, with column indices b_1, \dots, b_s , and row indices i_{d-s+1}, \dots, i_d , it is the evaluation of the Plücker coordinate $p_{\underline{i}}$ at z , where $\underline{i} = (i_1, \dots, i_d)$ may be described as follows: $\{i_1, \dots, i_{d-s}\}$ is the complement of $\{b_1, \dots, b_s\}$ in $\{1, \dots, d\}$, and i_{d-s+1}, \dots, i_d are simply the row indices (again, the rows of A are indexed as $d+1, \dots, n$).

13.10. Evaluation of the Plücker coordinates on the opposite big cell in G/Q . Consider

$$f : G \rightarrow G/Q \hookrightarrow G/P_{a_1} \times \cdots \times G/P_{a_k} \hookrightarrow \mathbf{P}_1 \times \cdots \times \mathbf{P}_k,$$

where $\mathbf{P}_t = \mathbb{P}(\wedge^{a_t} \mathbb{V})$. Denoting the restriction of f to O^- also by just f , we obtain an embedding $f : O^- \hookrightarrow \mathbf{P}_1 \times \cdots \times \mathbf{P}_k$, O^- having been identified with the opposite big cell in G/Q . For $z \in O^-$, the multi-Plücker coordinates of $f(z)$ are simply all the $a_t \times a_t$ minors of z with column indices $\{1, \dots, a_t\}$, $1 \leq t \leq k$.

13.11. Equations defining multicones over Schubert varieties in G/Q . Let $X_Q(w) \subset G/Q$. Denoting R, R_w as in §??, the kernel of the restriction map $R \rightarrow R_w$ is generated by the kernel of $R_1 \rightarrow (R(w))_1$; but now, in view of Proposition ??, this kernel is the span of

$$\{p_{\underline{i}} \mid \underline{i} \in I_{d,n}, d = a_1, \dots, a_k, w^{(d)} \not\geq \underline{i}\},$$

where $w^{(d)}$ is the d -tuple corresponding to the Schubert variety which is the image of $X_Q(w)$ under the projection $G/Q \rightarrow G/P_{a_t}$, $1 \leq t \leq k$.

13.12. Ideal of the opposite cell in $X(w)$. Let us denote $B^{-e_{\text{id},Q}} \cap X_Q(w)$ by just A_w . Then as in §??, we identify $B^{-e_{\text{id},Q}}$ with the unipotent subgroup O^- generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$, and consider A_w as a closed subvariety of O^- . In view of §??, we obtain

Proposition 13.13. *The ideal defining A_w in O^- is generated by*

$$\{p_{\underline{i}} \mid \underline{i} \in I_{d,n}, d = a_1, \dots, a_k, w^{(d)} \not\geq \underline{i}\}$$

(here, for $z \in O^-$, $p_{\underline{i}}(z)$ is as in §??).

14. SINGULAR LOCI OF CERTAIN LADDER DETERMINANTAL
VARIETIES

In this section we determine the singular loci of certain ladder determinantal varieties. Viewing $X_{2,n}$ as a ladder determinantal variety, we prove the conjecture of Section ??, for the case $d = 2$.

We assume $d = 2$, and view $\mathcal{L} = \{(\rangle, \mid) \mid \infty \leq \rangle < \mid \leq \backslash\}$ as being contained in $Y = \{(i, j) \mid 1 \leq i, j \leq n\}$. A Schubert variety $X(\tau)$ in $G_{2,n}$ degenerates to $X(\mathcal{L}_\tau)$, where $\mathcal{L}_\tau = \{\theta \in \mathcal{L} \mid \theta \leq \tau\} \subset \mathcal{L}$ (cf. [?]). The equations defining the variety $X(\mathcal{L}_\tau)$ are precisely the 2 minors of Y which are contained in \mathcal{L}_τ . Now we look at a more general type of varieties.

14.1. Let $Y = \{(b, a) \mid 1 \leq b \leq n, 1 \leq a \leq n\}$. Given $1 \leq b_1 < \cdots < b_l < n, 1 < a_1 < \cdots < a_l \leq n$, we consider the subset L of Y , defined by

$$L = \{(b, a) \mid \text{there exists } 1 \leq i \leq l \text{ such that } b_i \leq b \leq n, 1 \leq a \leq a_i\}.$$

We call L an *one-sided ladder* in Y , defined by the *outer corners* $\alpha_i = (b_i, a_i)$, $1 \leq i \leq l$ (see Figure 1). For $1 \leq i \leq l$, let L_i be the subset of Y defined by

$$L_i = \{(b, a) \mid b_i \leq b \leq n, 1 \leq a \leq a_i\}.$$

Clearly, $L = \cup_{i=1}^l L_i$.

We also view Y as the generic $n \times n$ matrix $(x_{ba})_{1 \leq b, a \leq n}$, and we say that $x_{ba} \in L$ if $(b, a) \in L$. Let $k[L]$ denote the polynomial ring $k[x_{ba} \mid x_{ba} \in L]$, and let $\mathbb{A}(\mathbb{L}) = \mathbb{A}^{\#L}$ be the associated affine space. Let $I(L)$ be the ideal in $k[L]$ generated by the 2 minors of Y which are contained in L , and $X(L) \subset \mathbb{A}(\mathbb{L})$ the variety defined by the ideal $I(L)$. A 2 minor in Y formed with two consecutive rows and two consecutive columns will be called a *solid minor*. We have (cf. [?]):

Lemma 14.2. *The codimension of $X(L)$ in $\mathbb{A}(\mathbb{L})$ is equal to the number of solid 2 minors in L .*

Theorem 14.3. *Let L be an one-sided ladder in Y defined by the outer corners $\alpha_i = (b_i, a_i)$, with $a_i \geq 2$ and $b_i \leq n - 1$, $1 \leq i \leq l$. For each $1 \leq i \leq l$, let $V_i = \{P = (P_\alpha)_{\alpha \in L} \in X(L) \mid P_\alpha = 0 \text{ for all } \alpha \in L_i\}$. Then the irreducible components of $\text{Sing } X(L)$ are precisely V_i , $1 \leq i \leq l$.*

Proof. First, we prove that $V_i \subset \text{Sing } X(L)$, for all $1 \leq i \leq l$. Fix $1 \leq i \leq l$, and let $P = (P_\alpha)_{\alpha \in L} \in V_i$. Let \mathcal{J} be the Jacobian matrix of $X(L)$. Then the rows of \mathcal{J} are indexed by $\{M \mid M \text{ is a 2 minor of } Y \text{ contained in } L\}$, and the columns are indexed by $\{x_\alpha \mid \alpha \in L\}$. Let \mathcal{J}_P be the Jacobian

matrix evaluated at P . Then the (M, x_α) -th entry in $\mathcal{J}_{\mathcal{P}}$ is equal to $\pm P_{\alpha'}$, where α' is the element in M which is neither in the row, nor in the column containing α , if x_α appears in M , and 0 otherwise. Since $P \in V_i$, the row indexed by a minor involving $x_{\alpha_i} = x_{(b_i, a_i)}$ is 0. Also, the column indexed by x_{α_i} is 0.

Let \mathcal{J}' be the matrix obtained from $\mathcal{J}_{\mathcal{P}}$ by deleting the column indexed by x_{α_i} , and the rows indexed by minors involving x_{α_i} . Then

$$\text{rank } \mathcal{J}_{\mathcal{P}} = \text{rank } \mathcal{J}', \quad (1)$$

since \mathcal{J}' is obtained from $\mathcal{J}_{\mathcal{P}}$ by deleting zero rows and columns.

Now consider the one-sided ladder L' obtained from L by deleting the element α_i . Let $P' = (P_\alpha)_{\alpha \in L'}$. Then $P' \in X(L')$, and \mathcal{J}' is the Jacobian matrix of $X(L')$ evaluated at P' . Thus

$$\text{rank } \mathcal{J}' \leq \text{codim}_{\mathbb{A}(L')} \mathcal{X}(\mathcal{L}'). \quad (2)$$

By hypothesis, $a_i \geq 2$ and $b_i \leq n - 1$, and hence there exist at least one solid minor in L not contained in L' , while every solid minor in L' is also contained in L . Thus, using Lemma ??, we deduce that

$$\text{codim}_{\mathbb{A}(L')} X(L') < \text{codim}_{\mathbb{A}(L)} X(L). \quad (3)$$

Using (1), (2) and (3), we deduce that

$$\text{rank } \mathcal{J}_{\mathcal{P}} < \text{codim}_{\mathbb{A}(L)} \mathcal{X}(\mathcal{L}),$$

i.e. $P \in \text{Sing } X(L)$.

Next we prove that $\text{Sing } X(L) \subset \cup_{i=1}^l V_i$. Let $P \in X(L) \setminus \cup_{i=1}^l V_i$. For each $1 \leq i \leq l$, we fix an element $\beta_i \in L_i$ such that $P_{\beta_i} \neq 0$. Let \mathcal{C} be the set obtained from L by deleting the elements appearing either in the first column, or in the last row of L . Then $\#\mathcal{C}$ is equal to the number of solid 2 minors in L , and by Lemma ??, we have

$$\#\mathcal{C} = \text{codim}_{\mathbb{A}(L)} \mathcal{X}(\mathcal{L}).$$

We have $\mathcal{C} = \dot{\cup}_{\gamma=\infty}^{\uparrow} \mathcal{C}_\gamma$, where $\mathcal{C}_\gamma = \mathcal{C} \cap (\mathcal{L}_\gamma \setminus \mathcal{L}_{\gamma+\infty})$, for $1 \leq i \leq l$, $L_{l+1} = \emptyset$. Let \mathcal{T}_\uparrow be the set of elements in L_l not in the row or the column of β_l . Clearly, $\#\mathcal{T}_\uparrow = \#\mathcal{C}_\uparrow$. By (decreasing) induction on i , suppose that, for some i , $1 < i \leq l$, the sets $\mathcal{T}_\gamma, \dots, \mathcal{T}_\uparrow$ have been constructed, such that

- (1) _{i} $\mathcal{T}_\uparrow \subset \mathcal{L}_\uparrow$, $i \leq j \leq l$,
- (2) _{i} the sets $\mathcal{T}_\gamma, \dots, \mathcal{T}_\uparrow$ are pairwise disjoint,
- (3) _{i} $\#\mathcal{T}_\uparrow = \#\mathcal{C}_\uparrow$, $i \leq j \leq l$,
- (4) _{i} \mathcal{T}_\uparrow contains no elements appearing in the column or in the row of β_j , $i \leq j \leq l$,
- (5) _{i} there exists a row in L_i not containing any element from $\mathcal{T}_\gamma \cup \dots \cup \mathcal{T}_\uparrow$.

We define the set $\mathcal{T}_{\gamma-\infty}$ as follows. If $\beta_{i-1} \notin L_{i-1} \setminus L_i$, then $\mathcal{T}_{\gamma-\infty}$ is obtained from $L_{i-1} \setminus L_i$ by deleting the elements in the column of β_{i-1} . If $\beta_{i-1} \in L_{i-1} \setminus L_i$, then $\mathcal{T}_{\gamma-\infty}$ is obtained as follows. Choose a row R_i in L_i , as given by (5)_{*i*}. We set $\mathcal{T}_{\gamma-\infty} = (\mathcal{L}_{\gamma-\infty} \setminus \mathcal{L}_{\gamma}) \cup \mathcal{R}_{\gamma} \setminus \mathcal{A}_{\beta_{i-1}}$, where $\mathcal{A}_{\beta_{i-1}}$ is the set of elements of L in the row and column of β_{i-1} . Clearly, $\mathcal{T}_{\gamma-\infty} \subset \mathcal{L}_{\gamma-\infty}$, the sets $\mathcal{T}_{\gamma-\infty}, \mathcal{T}_{\gamma}, \dots, \mathcal{T}_{\uparrow}$ are pairwise disjoint, $\#\mathcal{T}_{\gamma-\infty} = \#\mathcal{C}_{\gamma-\infty}$, and $\mathcal{T}_{\gamma-\infty}$ does not contain any element in the row or the column of β_{i-1} . Also, there exists a row in L_{i-1} which does not contain any element from $\mathcal{T}_{\gamma-\infty} \cup \mathcal{T}_{\gamma} \cup \dots \cup \mathcal{T}_{\uparrow}$. Hence the sets $\mathcal{T}_{\gamma-\infty}, \mathcal{T}_{\gamma}, \dots, \mathcal{T}_{\uparrow}$ satisfy (1)_{*i-1*} – (5)_{*i-1*}. Thus we obtain pairwise disjoint sets $\mathcal{T}_j \subset \mathcal{L}_j$, $1 \leq j \leq l$, such that $\#\mathcal{T}_j = \#\mathcal{C}_j$, and \mathcal{T}_j does not contain any element in the row or column of β_j . Let $\mathcal{T} = \bigcup_{j=-\infty}^{\uparrow} \mathcal{T}_j$. Then $\#\mathcal{T} = \#\mathcal{C}$.

For $\tau \in \mathcal{T}_j$, $1 \leq j \leq l$, let M^τ be the 2 minor determined by τ and β_j . Clearly, $M^\tau \neq M^{\tau'}$ for $\tau, \tau' \in \mathcal{T}$, $\tau \neq \tau'$. Let \mathcal{J}' be the submatrix of $\mathcal{J}_{\mathcal{P}}$ given by rows indexed by M^τ 's and the columns indexed by x_τ 's, with $\tau \in \mathcal{T}$. We index the rows and columns of \mathcal{J}' by the elements in \mathcal{T} , and we arrange them increasingly, with respect to the lexicographic order in Y (namely, $(b, a) > (b', a')$ if $b > b'$, or $b = b'$, $a > a'$).

Let us fix $\tau \in \mathcal{T}$, say $\tau \in \mathcal{T}_j$, $1 \leq j \leq l$. Since τ is the only entry in M^τ which belongs to \mathcal{T} , we deduce that in the τ -th row of \mathcal{J}' all the entries are zero, except the one in the τ -th column, which is equal to $\pm P_{\beta_j}$, and hence it is nonzero. Thus the matrix \mathcal{J}' is diagonal, with nonzero diagonal entries. Therefore its rank is equal to its size, which is $\#\mathcal{T}$. Since $\#\mathcal{T}$ is equal to $\#\mathcal{C}$, and hence equal to the codimension of $X(L)$, we deduce that

$$\text{rank } \mathcal{J}' = \text{codim}_{\mathbb{A}(\mathbb{L})} \mathcal{X}(\mathcal{L}). \quad (4)$$

Since $\text{rank } \mathcal{J}' \leq \text{rank } \mathcal{J}_{\mathcal{P}} \leq \text{codim}_{\mathbb{A}(\mathbb{L})} \mathcal{X}(\mathcal{L})$, (4) implies that

$$\text{rank } \mathcal{J}_{\mathcal{P}} = \text{codim}_{\mathbb{A}(\mathbb{L})} \mathcal{X}(\mathcal{L}),$$

i.e. $P \notin \text{Sing } X(L)$. Therefore we conclude that

$$\text{Sing } X(L) = \bigcup_{i=1}^l V_i.$$

Let $\mathcal{L}_{\gamma} = \mathcal{L} \setminus \mathcal{L}_{\gamma}$, $1 \leq i \leq l$. Then \mathcal{L}_{γ} is a distributive lattice, and V_i is identified with $X(\mathcal{L}_{\gamma})$. Using §??, we deduce that V_i is irreducible, $1 \leq i \leq l$. The fact that $V_i \not\subset V_j$ for $i \neq j$ is obvious. This completes the proof of the theorem. \square

The following theorem shows the validity of the Conjecture stated in Section ??, for the case $d = 2$.

Theorem 14.4. *Let $\mathcal{L}_{\rangle\infty} \subset \mathcal{I}_{\infty, \setminus}$, $1 \leq i \leq n-3$, be as in Section ???. Then the irreducible components of $\text{Sing } X_{2,n}$ are precisely $X(\mathcal{L}_{\rangle\infty})$, $1 \leq i \leq n-3$.*

Proof. First observe that $\phi_1 = (1, n)$ is the only join and meet-irreducible element of Type II in $I_{2,n}$, and the join and meet-irreducible elements of Type I are $\tau_i = (i+1, i+2)$, $0 \leq i \leq n-2$ (note that $\tau_0 = \widehat{0}$, $\tau_{n-2} = \widehat{1}$). We have $X_{2,n} = X(L) \times \mathbb{A}^{\neq}$, where $L = I_{2,n} \setminus \{\widehat{0}, \widehat{1}\}$. Using Theorem ??? for the ladder L , we obtain that the irreducible components of $X(L)$ are V_i , $1 \leq i \leq n-3$ (V_i being as defined in the statement of the Theorem). Thus the irreducible components of $X_{2,n}$ are precisely $V_i \times \mathbb{A}^{\neq}$, $1 \leq i \leq n-3$. It is easily seen that $X(\mathcal{L}_{\rangle\infty}) = \mathcal{V}_{\rangle} \times \mathbb{A}^{\neq}$, $1 \leq i \leq n-3$, and the result follows from this. \square

Remark 14.5. We have

$$\text{codim}_{X(L)} V_i = b_{i+1} - b_i + a_i - a_{i-1} + 1, \quad 1 \leq i \leq l$$

(here, $a_0 = 0$, $b_{l+1} = n$). In particular, taking $L = I_{2,n} \setminus \{\widehat{0}, \widehat{1}\}$, we deduce that $\text{Sing } X_{2,n}$ is of pure codimension three in $X_{d,n}$.

15. RELATIONSHIP BETWEEN LADDER DETERMINANTAL VARIETIES AND SCHUBERT VARIETIES

Let $Y = (x_{ba})$, $1 \leq b, a \leq n$ be a matrix of variables, and $L \subset Y$ an one-sided ladder defined by the outer corners $(b_1, a_1), \dots, (b_l, a_l)$, with $1 \leq b_1 < \dots < b_l < n$, $1 < a_1 < \dots < a_l \leq n$. We also assume that n is large enough such that L lies below the main diagonal of Y , i.e. $b_i > a_i$ for all $1 \leq i \leq l$.

Let $G = SL(n)$, and consider the parabolic subgroup $Q = P_{a_1} \cap \dots \cap P_{a_l}$ in G . Let O^- be the opposite big cell in G/Q . Let $X(L) \subset \mathbb{A}(\mathbb{L})$ be the variety defined by the vanishing of the 2 minors in L , as in Section ???. Let H be the one-sided ladder defined by the outer corners $(a_i + 1, a_i)$, $1 \leq i \leq l$, and let Z be the variety in $\mathbb{A}(\mathbb{H}) \simeq \mathbb{O}^-$ defined by the vanishing of the 2 minors in L . Note that $Z \simeq X(L) \times \mathbb{A}(\mathbb{H} \setminus \mathbb{L})$.

Let $Y^- = (y_{ba})$, $1 \leq b, a \leq n$, where

$$y_{ba} = \begin{cases} x_{ba}, & \text{if } (b, a) \in H \\ 1, & \text{if } b = a \\ 0, & \text{otherwise.} \end{cases}$$

Note that, given $\tau \in W^{a_i}$, $1 \leq i \leq l$, the function $p_{\tau}|_{O^-}$ represents the determinant of the submatrix T of Y^- whose row indices are $\{\tau(1), \dots, \tau(a_i)\}$, and column indices are $\{1, \dots, a_i\}$.

We shall now define an element $w \in W^Q$ such that Z gets identified with the opposite cell in $X(w)$ (see also [?]). We define $w \in W^Q$ by specifying $w^{(i)} \in W^{a_i}$, where $\pi_i(X(w)) = X(w^{(i)})$ under the projection $\pi_i : G/Q \rightarrow G/P_{a_i}$, $1 \leq i \leq l$.

Define $w^{(a_i)}$, $1 \leq i \leq l$, inductively, as the maximal elements in W^{a_i} with the following properties:

- (1) $w^{(a_i)}(a_i - 1) = b_i - 1$,
- (2) if $i > 1$, then $w^{(a_{i-1})} \subset w^{(a_i)}$.

Note that $w \in W^Q$, and it is maximal for the property

$$w^{(a_i)}(a_i - 1) < b_i,$$

for all $1 \leq i \leq l$.

Theorem 15.1. *The variety Z identifies with the opposite cell in $X(w)$, i.e. $Z = X(w) \cap O^-$ (scheme theoretically).*

Proof. Let f be a generator of $I(Z)$, i.e. $f = \det M$, for some 2×2 matrix M contained in L . Let $r_1 < r_2$ (resp. $c_1 < c_2$) be the row (resp. column) indices of M . Let $1 \leq i \leq l$ smallest such that M is contained in L_i . Thus $r_1, r_2 \geq b_i > a_i$, $c_2 > a_{i-1}$ (here $a_{i-1} = 0$ if $i = 1$). Let $\tau = \{1, \dots, a_i\} \setminus \{c_1, c_2\} \cup \{r_1, r_2\}$. Then $\tau \in W^{a_i}$, and $p_\tau|_{O^-} = \det T$, where T is the $a_i \times a_i$ submatrix of Y^- whose row indices are $\{\tau(1), \dots, \tau(a_i)\}$, and column indices are $\{1, \dots, a_i\}$. Using Laplace expansion with respect to the last two rows of T , we obtain

$$\det T = \sum \pm \det N_{c', c''} \det M_{c', c''}, \quad (*)$$

the sum being taken over all subsets $\{c', c''\} \subset \{1, \dots, a_i\}$, $c' \neq c''$, where $N_{c', c''}$ is the $(a_i - 2) \times (a_i - 2)$ submatrix of Y^- whose row indices are $\{1, \dots, a_i\} \setminus \{c_1, c_2\}$ and column indices are $\{1, \dots, a_i\} \setminus \{c', c''\}$, and $M_{c', c''}$ is the 2×2 submatrix of Y^- whose row indices are $\{r_1, r_2\}$ and column indices are $\{c', c''\}$. Note that $M_{c_1, c_2} = M$, and N_{c_1, c_2} is a lower triangular matrix, with all diagonal entries equal to 1, and hence $\det M$ appears in (*), and its coefficient is ± 1 . Also note that $N_{c', c''}$ is obtained from N_{c_1, c_2} by replacing the columns with indices c', c'' by the columns with indices c_1, c_2 .

By decreasing induction with respect to the index c_1 of the first column of M , we prove that $f = \det M$ can be written in the form $f = \sum g_\phi p_\phi|_{O^-}$, with $\phi \in W^{a_i}$, $\{a_i + 1, \dots, n\} \cap \{\phi(1), \dots, \phi(a_i)\} = \{r_1, r_2\}$, and $g_\phi \in k[H]$.

If $c_1 > a_{i-1}$, then for $\{c', c''\} \neq \{c_1, c_2\}$ we have $\det N_{c', c''} = 0$, since at least one of c_1, c_2 is an index for a column in $N_{c', c''}$, and all entries of this column are 0. Thus, in this case (*) reduces to $\det T =$

$\pm \det M$, i.e. $\det M = \pm p_\tau|_{O^-}$, with $\tau \in W^{a_i}$ such that $\{a_i + 1, \dots, n\} \cap \{\tau(1), \dots, \tau(a_i)\} = \{r_1, r_2\}$.

Let $c_1 \leq a_{i-1}$ such that the above statement is true for $c_1 + 1$. First we observe that if $c_2 \notin \{c', c''\}$, then $\det N_{c', c''} = 0$, since in this case c_2 is an index for a column in $N_{c', c''}$, and all entries of this column are 0. Let then $c_2 \in \{c', c''\}$, and let $\{c', c''\} = \{c, c_2\}$. Then N_{c, c_2} is obtained from N_{c_1, c_2} by replacing the column with index c by the column with index c_1 . If $c < c_1$, then N_{c, c_2} is still lower triangular, but the diagonal entry in the column with index c_1 is 0, and hence $\det N_{c, c_2} = 0$. Therefore we obtain

$$\det T = \pm \det M + \sum_{c \in \{c_1+1, \dots, a_i\} \setminus \{c_2\}} \pm \det N_{c, c_2} \det M_{c, c_2},$$

and hence

$$f = \det M = \pm p_\tau|_{O^-} + \sum_{c \in \{c_1+1, \dots, a_i\} \setminus \{c_2\}} \pm \det N_{c, c_2} \det M_{c, c_2}.$$

Using induction hypothesis for M_{c, c_2} , we obtain $f = \sum g_\phi p_\phi|_{O^-}$, with $\phi \in W^{a_i}$ such that $\{a_i + 1, \dots, n\} \cap \{\phi(1), \dots, \phi(a_i)\} = \{r_1, r_2\}$, and $g_\phi \in k[H]$. In particular $\phi(a_i - 1) = r_1$. Since $r_1 \geq b_i$, we deduce that $\phi(a_i - 1) \geq b_i$. We have $w^{(a_i)}(a_i - 1) = b_i - 1$, and hence $\phi(a_i - 1) > w^{(a_i)}(a_i - 1)$. This shows that $\phi \not\leq w^{(a_i)}$, and hence $p_\phi|_{O^-} \in I(X(w) \cap O^-)$. Therefore $f \in I(X(w) \cap O^-)$.

Let now g be a generator of the ideal $I(X(w) \cap O^-)$, i.e. $g = p_\tau|_{O^-}$, with $\tau \in W^{a_i}$ for some $1 \leq i \leq l$, such that $\tau \not\leq w^{a_i}$ (cf. §??). Since $w^{(a_i)}$ consists of several blocks of consecutive integers ending with $b_t - 1$ at the $(a_t - 1)$ -th place, for some t 's in $\{1, \dots, i\}$, and a last block ending with n at the a_i -th place, it follows that there exists a $t \in \{1, \dots, i\}$ such that $\tau(a_t - 1) \geq b_t$. As above, the function $p_\tau|_{O^-}$ represents the determinant of the submatrix T of Y^- whose row indices are $\{\tau(1), \dots, \tau(a_i)\}$, and column indices are $\{1, \dots, a_i\}$. Using Laplace expansion with respect to the first a_t columns, we have $\det T = \sum_p \det A_p \det B_p$, where A_p (resp. B_p) is an $a_t \times a_t$ (resp. $(a_i - a_t) \times (a_i - a_t)$) matrix. Clearly, all the column indices of A_p are $\leq a_t$, and since $\tau(a_t - 1) \geq b_t$, at least 2 row indices of A_p are $\geq b_t$. Using Laplace expansion for A_p with respect to 2 rows whose indices are $\geq b_t$, we obtain $\det A_p = \sum_q \det C_q \det D_q$, where C_q (resp. D_q) is a 2 (resp. $a_t - 2$) minor, with C_q contained in $L_t \subset L$. This shows that $p_\tau|_{O^-} \in I(Z)$. This completes the proof. \square

Corollary 15.2. *The variety $X(L)$ is normal, Cohen-Macaulay, and has rational singularities.*

This follows from Theorem ??, and the fact that Schubert varieties are normal, Cohen-Maculay, and have rational singularities (cf. [?], [?], and [?]).

Let us fix $j \in \{1, \dots, l\}$. We shall now define $\theta_j \in W^Q$ such that $Z_j = V_j \times \mathbb{A}(\mathbb{H} \setminus \mathbb{L})$ gets identified with the opposite cell in $X(\theta_j)$.

For $i < j$, let $\theta_j^{(a_i)} = w^{(a_i)} \setminus \{n\} \cup \{b_j - 1\}$.

For $i = j$, let $\theta_j^{(a_j)} = w^{(a_j)} \setminus \{n\} \cup \{x_j\}$, where x_j is the maximal element in $\{1, \dots, b_j - 1\} \setminus w^{(a_j)}$.

For $i > j$, let

$$\theta_j^{(a_i)} = \begin{cases} w^{(a_i)}, & \text{if } x_j \in w^{(a_i)} \\ w^{(a_i)} \setminus \{y_i\} \cup \{x_j\}, & \text{if } x_j \notin w^{(a_i)}, \end{cases}$$

where y_i is the minimal element in $w^{(a_i)} \setminus \theta_j^{(a_{i-1})}$.

Lemma 15.3. *We have $\theta_j \leq w$. Further, θ_j is the maximal element $\tau \in W^Q$, $\tau \leq w^{max}$, such that $n \notin \tau^{(a_j)}$, and $n \in \tau^{(a_{j+1})}$.*

The assertion is clear from the definition of θ_j .

Theorem 15.4. *The subvariety $Z_j \subset Z$ gets identified with the opposite cell in $X(\theta_j)$, i.e. $Z_j = X(\theta_j) \cap O^-$ (scheme theoretically).*

Proof. Let f be a generator of $I(Z_j)$. If $f \in I(Z)$, then, in view of Theorem ?? we have $f \in I(X(w) \cap O^-) \subset I(X(\theta_j) \cap O^-)$ (since $w \geq \theta_j$), and there is nothing to prove. Assume that $f \notin I(Z)$; then $f = x_\alpha$, for some $\alpha = (b, a) \in L_j$. Then f can be written as $p_\phi|_{O^-}$, with $\phi \in W^{a_j}$, such that $\{b\} = \{a_j + 1, \dots, n\} \cap \{\phi(1), \dots, \phi(a_j)\}$, and $\{a\} = \{1, \dots, a_j\} \setminus \{\phi(1), \dots, \phi(a_i)\}$. Thus $\phi(a_j) = b$, and since $\alpha \in L_j$, we have $b \geq b_j$. Therefore $\phi(a_j) \geq b_j$. But $\theta_j^{(a_j)}(a_j) = b_j - 1$, and hence $\phi(a_j) > \theta_j^{(a_j)}(a_j)$. This shows that $\phi \not\leq \theta_j^{(a_j)}$, and therefore $f \in I(X(\theta_j) \cap O^-)$.

Let now g be a generator of the ideal $I(X(\theta_j) \cap O^-)$, i.e. $g = p_\tau|_{O^-}$, with $\tau \in W^{(a_i)}$, for some $1 \leq i \leq l$, such that $\tau \not\leq \theta_j^{(a_i)}$.

First assume that $i \leq j$. Then $\theta_j^{(a_i)}$ consists of several blocks of consecutive integers ending with $b_t - 1$ at the $(a_t - 1)$ -th place, for some t 's in $\{1, \dots, i - 1\}$, and a last block ending with $b_j - 1$ at the a_i -th place. The condition $\tau \not\leq \theta_j^{(a_i)}$ implies that either there exists $t \in \{1, \dots, i - 1\}$ such that $\tau(a_t - 1) \geq b_t$, or $\tau(a_i) \geq b_j$. In the first case we have $\tau \not\leq w^{(a_i)}$, and hence $p_\tau|_{O^-} \in I(X(w) \cap O^-) = I(Z) \subset I(Z_j)$. Suppose now that $\tau(a_i) \geq b_j$. The function $p_\tau|_{O^-}$ represents the determinant of the submatrix T of Y^- whose row indices are $\{\tau(1), \dots, \tau(a_i)\}$, and column indices are $\{1, \dots, a_i\}$. Obviously, all column indices are $\leq a_j$.

On the other hand, since $\tau(a_i) \geq b_j$, the last row of T is contained in L_i , and expanding T along this row, we deduce $p_\tau|_{O^-} \in I(Z_j)$.

Assume now that $i > j$. If $\theta_j^{(a_i)} = w^{(a_i)}$, then $\tau \not\leq w^{(a_i)}$, and hence $p_\tau|_{O^-} \in I(X(w) \cap O^-) = I(Z) \subset I(Z_j)$, and there is nothing to prove. Suppose that $\theta_j^{(a_i)} \neq w^{(a_i)}$. Then $\theta_j^{(a_i)}$ consists of several blocks of consecutive integers ending with $b_t - 1$ at the $(a_t - 1)$ -th place, for some t 's in $\{1, \dots, i\} \setminus \{j\}$, a block ending with $b_j - 1$ at the a_j -th place, and a last block ending with n at the a_i -th place. The condition $\tau \not\leq \theta_j^{(a_i)}$ implies that either there exists $t \in \{1, \dots, i\} \setminus \{j\}$ such that $\tau(a_t - 1) \geq b_t$, or $\tau(a_j) \geq b_j$. In the first case we have $\tau \not\leq w^{(a_i)}$, and hence $p_\tau|_{O^-} \in I(X(w) \cap O^-) = I(Z) \subset I(Z_j)$. In the second case, the function $p_\tau|_{O^-}$ represents the determinant of the submatrix T of Y^- whose row indices are $\{\tau(1), \dots, \tau(a_i)\}$, and column indices are $\{1, \dots, a_i\}$. Using Laplace expansion with respect to the first a_j columns, we have $\det T = \sum_p \det A_p \det B_p$, where A_p (resp. B_p) is an $a_j \times a_j$ (resp. $(a_i - a_j) \times (a_i - a_j)$) matrix. Clearly, all the column indices of A_p are $\leq a_j$, and since $\tau(a_j) \geq b_j$, at least one row index of A_p is $\geq b_j$. Using Laplace expansion for A_p with respect to a row with index $\geq b_j$, we obtain $\det A_p = \sum_q C_q \det D_q$, where C_q 's are entries of a row of A_p , contained in $L_j \subset L$. This shows that $p_\tau|_{O^-} \in I(Z_j)$. \square

16. VALIDITY OF THE CONJECTURE IN [17]

Let $G = SL(n)$. In this section we prove the conjecture in [?] on the irreducible components of the singular locus for a class of Schubert varieties, namely the pull-backs $\pi^{-1}(X_Q(w))$ under $\pi : G/B \rightarrow G/Q$, where w and Q are as in Section ???. We have $\pi^{-1}(X_Q(w)) = X_B(w^{\max})$, where w^{\max} , as a permutation, is given by $w^{(a_1)}$ arranged in descending order, followed by $w^{(a_2)} \setminus w^{(a_1)}$, etc.. We shall refer to the set $w^{(a_i)} \setminus w^{(a_{i-1})}$, $1 \leq i \leq l+1$, arranged in descending order, as the i -th block in w^{\max} (here, $w^{(a_0)} = \emptyset$, and $w^{(a_{l+1})}$ is the set $\{1, \dots, n\} \setminus w^{(a_l)}$ arranged in descending order).

Remark 16.1. All of the entries in the i -th block in w^{\max} are $\leq b_i - 1$, $2 \leq i \leq l$.

Let $\eta \in W$. We shall denote $X_B(\eta)$ by just $X(\eta)$. We first recall the criterion given in [?] for $X(\eta)$ to be singular.

Theorem 16.2. *Let $\eta = (a_1 \dots a_n) \in \mathcal{S}_\setminus$. Then $X(\eta)$ is singular if and only if there exist i, j, k, m , $1 \leq i < j < k < m \leq n$ such that*

$$\text{either } a_k < a_m < a_i < a_j \text{ or } a_m < a_j < a_k < a_i.$$

16.3. **The set E_η .** We next recall the conjecture in [?] on the irreducible components of $\text{Sing } X(\eta)$.

Let $\eta = (a_1 \dots a_n) \in \mathcal{S}_\setminus$. Let E_η be the set of all $\tau' \leq \eta$ such that either 1) or 2) below holds.

- 1) There exist $i, j, k, m, 1 \leq i < j < k < m \leq n$, such that
 - (a) $a_k < a_m < a_i < a_j$
 - (b) if $\tau' = (b_1 \dots b_n)$, then there exist $i', j', k', m', 1 \leq i' < j' < k' < m' \leq n$ such that $b_{i'} = a_k, b_{j'} = a_i, b_{k'} = a_m, b_{m'} = a_j$
 - (c) if τ (resp. η') is the element obtained from η (resp. τ') by replacing a_i, a_j, a_k, a_m respectively by a_k, a_i, a_m, a_j (resp. $b_{i'}, b_{j'}, b_{k'}, b_{m'}$ respectively by $b_{j'}, b_{m'}, b_{i'}, b_{k'}$), then $\tau' \geq \tau$ and $\eta' \leq \eta$.
- 2) There exist $i, j, k, m, 1 \leq i < j < k < m \leq n$, such that
 - (a) $a_m < a_j < a_k < a_i$
 - (b) if $\tau' = (b_1 \dots b_n)$, then there exist $i', j', k', m', 1 \leq i' < j' < k' < m' \leq n$ such that $b_{i'} = a_j, b_{j'} = a_m, b_{k'} = a_i, b_{m'} = a_k$
 - (c) if τ (resp. η') is the element obtained from η (resp. τ') by replacing a_i, a_j, a_k, a_m respectively by a_j, a_m, a_i, a_k (resp. $b_{i'}, b_{j'}, b_{k'}, b_{m'}$ respectively by $b_{k'}, b_{i'}, b_{m'}, b_{j'}$), then $\tau' \geq \tau$ and $\eta' \leq \eta$.

Conjecture . *The singular locus of $X(\eta)$ is equal to $\cup_\lambda X(\lambda)$, where λ runs over the maximal (under the Bruhat order) elements of E_η .*

16.4. Let $\eta = (a_1 \dots a_n) \in \mathcal{S}_\setminus$. Let $\text{Sing } X(\eta) \neq \emptyset$. Let (a, b, c, d) be four distinct entries in $\{1, \dots, n\}$ such that $a < b < c < d$. An occurrence in η of the form d, b, c, a , where $d = a_i, b = a_j, c = a_k, a = a_m, i < j < k < m$, will be referred to as a *Type I bad occurrence in η* . An occurrence in η of the form (c, d, a, b) , where $c = a_i, d = a_j, a = a_k, b = a_m, i < j < k < m$, will be referred to as a *Type II bad occurrence in η* . Let (d, b, c, a) (resp. (c', d', a', b')) be a bad occurrence of Type I (resp. Type II), where $a < b < c < d$ (resp. $a' < b' < c' < d'$). Let θ, θ' be both $\leq w$. Further, let b, a, d, c (resp. a', c', b', d') appear in that order in θ (resp. θ'). By abuse of language, we shall refer to (b, a, d, c) (resp. (a', c', b', d')) as a bad occurrence in θ (resp. θ') corresponding to the bad occurrence (d, b, c, a) (resp. (c', d', a', b')) in η .

Lemma 16.5. *Let w be as in Section ???. Then any bad occurrence in $X(w^{\max})$ is of Type I.*

Proof. Let $w^{\max} = (a_1 \dots a_n)$. Assume that (c, d, a, b) is a bad occurrence of Type II in w^{\max} , where $a < b < c < d$. Clearly, c and d (resp. a and b) cannot both appear in the same block, in view of the description of w^{\max} . Let then c, d, a, b appear in the h -th, i -th, j -th, k -th blocks respectively, where $h < i \leq j < k$. This implies that $a < b < c < d \leq b_i - 1$ (cf. Remark ??). But now, a and b are both

$< b_i - 1$, and they both appear after $b_i - 1$; further, a appears before b in w^{\max} , which is not possible by the construction of w^{\max} (note that $a < b$). The required result follows from this. \square

Remark 16.6. Of course, there are several bad occurrences in w^{\max} of Type I. For example, fix some j , $1 \leq j \leq l$. Then $b_{j+1} - 1$ belongs to the $(j + 1)$ -th block of w^{\max} , and $b_{j+1} - 1 \geq b_j$ (here, if $j = l$, then $b_{j+1} - 1 = n - 1$). Let z_{j+1} be the smallest entry $\geq b_j$, in the $(j + 1)$ -th block of w^{\max} . Observe that

$$z_{j+1} = \begin{cases} y_{j+1}, & \text{if } x_j \notin w^{(a_{i+1})} \\ b_j, & \text{if } x_j \in w^{(a_{i+1})} \end{cases}$$

(here, x_j, y_{j+1} are as in Section ??). Note that if $x_j \in w^{(a_{i+1})}$, then b_j also belongs to $w^{(a_{i+1})}$. Take $d = n$, $b = b_j - 1$, $c = z_{i+1}$, $a = x_j$. Then d, b, c, a occur in the 1-st, j -th, $(j + 1)$ -th, m -th blocks respectively, where $m \geq j + 1$. This provides an example of a Type I bad occurrence in w^{\max} .

Lemma 16.7. *Let d, b, c, a be a bad occurrence in w^{\max} , where $a < b < c < d$. Assume that b belongs to the i -th block, for some i (note that $i \leq l$, since $b < c$). Then*

- (1) $b \leq b_i - 1$
- (2) $d = n$.

Proof. Let d, b, c, a occur in the h -th, i -th, j -th, m -th blocks respectively in w^{\max} , where $h \leq i < j \leq m$. First observe that $b < n$, since $b < c < d \leq n$. Hence, if $i = 1$, then $b \leq b_1 - 1$. If $i \geq 2$, again $b \leq b_i - 1$ (cf. Remark ??).

Claim . $d > b_i - 1$.

Proof. Assume that $d \leq b_i - 1$. Then assumption implies $c < b_i - 1$ (since $c < d$). Now both c and b are $< b_i - 1$, and b belongs to the i -th block in w^{\max} . This implies that c should occur before b , which is not possible. Hence our assumption is wrong, and the claim follows. \square

Note that the Claim in fact implies that $d = n$ (and $h = 1$). \square

Let $\theta \in SL(n)$ such that $\text{Sing } X(\theta) \neq \emptyset$.

Proposition 16.8. *The maximal elements in $E_{w^{\max}}$ are precisely θ_j^{\max} , $1 \leq i \leq l$ (here $E_{w^{\max}}$ is as in §??).*

Proof. We first observe that $\theta_j^{\max} \in E_{w^{\max}}$; for, corresponding to the bad occurrence $d = n$, $b = b_j - 1$, $c = z_{j+1}$, $a = x_j$ (cf. Remark ??), we have the bad occurrence (b, a, d, c) (note that b, a, d, c occur in that order in θ_j^{\max}). Let us denote θ_j^{\max} by τ' . Let w' (resp. τ) be the

element of $\mathcal{S}\setminus$ obtained from τ' (resp. w) by replacing b, a, d, c (resp. d, b, c, a) respectively by d, b, c, a (resp. b, a, d, c). Then clearly $\tau \leq \tau'$, and $w' \leq w$.

Let now $\tau' \in E_{w^{\max}}$. Then we have a bad occurrence in τ' , which has to be of the form (b, a, d, c) , $a < b < c < d$, corresponding to the occurrence (d, b, c, a) in w^{\max} (cf. Lemma ??). We have $d = n$ (cf. Lemma ??).

Claim . $d \notin \tau'^{(a_1)}$

Proof. Assume that $d \in \tau'^{(a_1)}$. This implies that $b, a \in \tau'^{(a_1)}$. Let w' be the element obtained from τ' by replacing b, a, d, c respectively by d, b, c, a . Now $w' \leq w$ (cf. §??) implies, in particular, that $w'^{(a_1)} \leq w^{(a_1)}$. Hence we obtain that b and c are both $\leq b_1 - 1$. Now in w^{\max} , we have, $c \notin w^{(a_1)}$, and c appears after b . But this is not possible (by the construction of w^{\max}). The claim now follows. \square

Now the claim implies that there exists an $r > 1$ such that $d \in \tau^{(a_r)}$, $d \notin \tau^{(a_{r-1})}$. Hence $\tau' \leq \theta_{r-1}^{\max}$ (cf. Lemma ??). The required result now follows from this \square

Theorem 16.9. *The conjecture ?? holds for $X(w^{\max})$.*

Proof. In view of Theorems ??, ??, and ?? $\text{Sing } X(\theta_j^{\max})$, $1 \leq j \leq l$ are precisely the irreducible components of $X(w^{\max})$. On the other hand, we have (cf. Proposition ??) that the maximal elements in $E_{w^{\max}}$ are precisely θ_j^{\max} , $1 \leq j \leq l$. Hence the irreducible components of $\text{Sing } X(w^{\max})$ are precisely $\{X(\theta) \mid \theta \in E_{w^{\max}}\}$. Thus the conjecture holds for $X(w^{\max})$. \square

REFERENCES

- [1] M. Aigner, *Combinatorial Theory*, Springer-Verlag, New York, 1979.
- [2] A. Björner, A. Garsia and R. Stanley, *An introduction to Cohen-Macaulay partially ordered sets*, I. Rival (editor), D. Reidel Publishing Company, 1982, 583-615.
- [3] A. Borel, *Linear Algebraic Groups*, second edition, Springer-Verlag, New York, 1991.
- [4] N. Bourbaki, *Groups et Algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968.
- [5] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms*, Springer-Verlag, New York, 1992.
- [6] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York, 1995.
- [7] D. Eisenbud and B. Sturmfels, *Binomial ideals*, preprint (1994).
- [8] W. Fulton, *Introduction to Toric Varieties*, Annals of Math. Studies **131**, Princeton U. P., Princeton N. J., 1993.

- [9] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties* (to appear in Transformation Groups).
- [10] N. Gonciulea and V. Lakshmibai, *Singular loci of ladder determinantal varieties and Schubert varieties* (in preparation)
- [11] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.
- [12] T. Hibi, *Distributive lattices, affine semigroup rings, and algebras with straightening laws*, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math. **11** (1987) 93-109.
- [13] H. Hiller, *Geometry of Coxeter groups*, Pitman Adv. pub. prog.
- [14] G. Kempf et al, *Toroidal Embeddings*, Lecture notes in Mathematics, NO. 339, Springer-Verlag, 1973.
- [15] G. Kempf and A. Ramanathan, *Multicones over Schubert varieties*, Inv. Math. **87** (1987), 353-363.
- [16] D. Glassbrenner and K. E. Smith, *Singularities of certain ladder determinantal varieties*, Journal of Pure and Applied Algebra, **100**, (1995), 59-75.
- [17] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in $SL(n)/B$* , Proc. Indian Acad. Sci. (Math. Sci.), **100**, (1990), 45-52.
- [18] V. Lakshmibai and J. Weyman, *Multiplicities of points in a minuscule G/P* , Adv. Math. **84** (1990), 179-208.
- [19] S. B. Mulay, *Determinantal loci and the flag variety*, Adv. Math. **74** (1989), 1-30.
- [20] S. Ramanan and A. Ramanathan, *Projective normality of Flag varieties and Schubert varieties*, Inv. Math. **79** (1985), 217-224.
- [21] A. Ramanathan, *Schubert varieties are arithmetically Cohen-Macaulay*, Inv. Math. **80** (1985), 283-294.
- [22] A. Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals*, Pub. Math. I.H.E.S. **65** (1987), 61-90.
- [23] B. Sturmfels, *Gröbner bases and convex polytopes*, Lecture notes from the New Mexico State University, preprint, 1995.
- [24] D. G. Wagner, *Singularities of toric varieties associated with finite distributive lattices*, preprint (1995).

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115

E-mail address: ngonciul@lynx.neu.edu

E-mail address: ngonciul@lynx.neu.edu, lakshmibai@neu.edu