ON TANGENT SPACES TO SCHUBERT VARIETIES

V. LAKSHMIBAI†

Abstract. We prove the results on the tangent spaces to Schubert varieties announced in [?] for G classical. We give two descriptions of the tangent space to a Schubert variety at *id*. The first description is in terms of the root system, and the second one in terms of multiplicities of certain weights in the fundamental representations of G.

Introduction

Let G be a semi simple, simply connected algebraic group over an algebraically closed field K of characteristic 0. Let T be a maximal torus in G, and W the Weyl group. Let R be the system of roots of G relative to T. Let B be a Borel subgroup of G, where $B \supset T$. Let S (resp.R^+) be the set of simple (resp. positive) roots of R relative to B. For $\alpha \in R$, let s_α be the reflection , and X_α the element of the Chevalley basis for $\mathfrak{g}\,(= \mathrm{Lie}G)$, corresponding to α. For $w \in W$, let us denote the point in G/B corresponding to the coset wB by e_w. Then the set of T-fixed points in G/B for the action given by left multiplication is presisely $\{e_w \mid w \in W\}$. For $w \in W$, let $X(w)$ denote the associated Schubert variety (the Zariski closure of Be_w in G/B). Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$, and $U^+(\mathfrak{g})$ the subalgebra of $U(\mathfrak{g})$ generated by $\{X_\alpha, \alpha \in S\}$.

For $\tau \in W$, let $T(w_0, e_\tau)$ denote the tangent space to G/B at e_τ (w_0 being the element of largest length in W). We have

$$T(w_0, e_\tau) := \oplus_{\ \beta\in\tau(R^+)}\ \mathfrak{g}_{-\beta}.$$

For $\tau \leq w$, let $T(w,\tau)$ be the Zariski tangent space to $X(w)$ at e_τ. Let

$$N(w,\tau) = \{\beta \in \tau(R^+) \mid X_{-\beta} \in T(w,\tau)\}.$$

Since $T(w, e_\tau)$ is a T-stable subspace of $T(w_0, e_\tau)$, we have that $T(w, e_\tau)$ is spanned by $\{X_{-\beta},\, \beta \in N(w,\tau)\}$.

For a dominant weight λ, let $V(\lambda)$ be the irreducible G-module with highest weight λ. Let us fix a highest weight vector $u(\lambda)$ in $V(\lambda)$. For $w \in W$, fix a representative n_w for w in $N_T(G)$ (the normalizer of

†Partially supported by NSF Grant DMS 9502942.

T in G), and set $u_w(\lambda) = n_w \cdot u(\lambda)$, $V_w(\lambda) = U^+(\mathfrak{g})\mathfrak{u}_{\mathfrak{w}}(\lambda)$ (note that $V_{w_0}(\lambda) = V(\lambda)$).

For $w \in W$, we define M_w, a subset of R^+ as follows, G being classical: (We index the roots as in [**?**].)

1. Let G be of type $\mathbf{A_n}$. Then $M_w = \{\beta \in R^+ \mid w \geq s_\beta\}$.
2. Let G be of type $\mathbf{C_n}$.
   (a) Let $\beta = \epsilon_i - \epsilon_j$, or $2\epsilon_i$. Then $\beta \in M_w \iff w \geq s_\beta$.
   (b) Let $\beta = \epsilon_i + \epsilon_j$, $i < j \leq n$. Then $\beta \in M_w \iff w \geq s_{\epsilon_i+\epsilon_j}$ or $s_{2\epsilon_i}$.
3. Let G be of type $\mathbf{B_n}$.
   (a) Let $\beta = \epsilon_i - \epsilon_j$, ϵ_n, or $\epsilon_i + \epsilon_n$. Then $\beta \in M_w \iff w \geq s_\beta$.
   (b) Let $\beta = \epsilon_i, i < n$. Then $\beta \in M_w \iff w \geq s_{\epsilon_i}$ or $s_{\epsilon_i+\epsilon_n}$.
   (c) Let $\beta = \epsilon_i + \epsilon_j, i < j < n$. Then $\beta \in M_w \iff w \geq s_{\epsilon_i+\epsilon_j}$ or $s_{\epsilon_i}s_{\epsilon_j+\epsilon_n}$.
4. Let G be of type $\mathbf{D_n}$.
   (a) Let $\beta = \epsilon_k - \epsilon_l$, or $\epsilon_i + \epsilon_j$, $j = n-1, n$. Then $\beta \in M_w \iff w \geq s_\beta$.
   (b) Let $\beta = \epsilon_i + \epsilon_j$, $i < j < n-1$.
   Then $\beta \in M_w \iff w \geq s_{\epsilon_i+\epsilon_j}$ or $s_{\epsilon_i-\epsilon_n}s_{\epsilon_i+\epsilon_n}s_{\epsilon_j+\epsilon_{n-1}}$.

We prove for G classical the following result: (cf. Theorems 3.4,4.6,5.7,6.8)

Theorem 1 . *Let $w \in W$. Let e denote the identity element in W. Then $N(w,e) = M_w$.*

In particular, we obtain a criterion for smoothness:

Theorem 2 . *Let $w \in W$. Then $X(w)$ is smooth if and only if $\#M_w = l(w)$.*

We further prove (cf. Theorems 3.5,5.9,6.10)

Theorem 3 . *Let G be of type $\mathbf{A_n}$, $\mathbf{B_n}$, or $\mathbf{D_n}$. Let $w \in W$, and $\beta \in R^+$. Then $\beta \in N(w,e)$ if and only if $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$, where $\omega_d, 1 \leq d \leq n$ are the fundamental weights of G and $m(\omega_d-\beta)$ (resp. $m_w(\omega_d-\beta)$) denotes the multiplicity of $\omega_d-\beta$ in $V(\omega_d)$ (resp.$V_w(\omega_d)$).*

Remark 0.1. It turns out that the above result is not true for Type $\mathbf{C}_n$ (see §4 for details).

Let $w \in W$, $\beta \in R^+$. Consider the following three conditions:

1. $w \geq s_\beta$.
2. $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$, n being the rank of G.
3. $\beta \in N(w,e)$.

We show (cf. Theorem **??**) that for G of Type $\mathbf{A_n}$, the above three conditions are equivalent for all $w \in W$, $\beta \in R^+$. For the types $\mathbf{B_n}$, $\mathbf{C_n}$, $\mathbf{D_n}$, we give precise relationships among the above three conditions (see sections 4,5,6 for details).

The above results are proved using the basis $\mathcal{B}_d$ (cf. [**?**]) for $V(\omega_d)$, $1 \leq d \leq l$, l being the rank of G, and a Theorem of Polo (cf. [**?**], Theorem 3.2). All of the results of this paper hold over arbitrary characteristics, since one knows (see [**?**], Corollary 4.1 for example) that $N(w,\tau)$ is independent of the base field; in Theorem 3 over an algebraically closed field K of arbitrary characteristic, one should replace $V(\omega_d)$ (resp.$V_w(\omega_d)$) by the corresponding Weyl (resp. Demazure) module.

The sections are organized as follows. In §1, we recall the above mentioned result of Polo. In §2, we recall the basis $\mathcal{B}_d$. In sections §3,4,5,6 we prove Theorems 1&3 for G of type $\mathbf{A}_n$, $\mathbf{C}_n$, $\mathbf{B}_n$, and $\mathbf{D}_n$ respectively.

The author is thankful to the referee for some useful comments.

1. THE TANGENT SPACE $T(w,\tau)$.

Let G be a semisimple and simply connected algebraic group defined over K. Let $T, B, W, S, R, R^+, e_w, X(w)$ etc be as in the Introduction. We have the well-known Bruhat decomposition

$$G/B = \dot{\cup}_{\{w \in W\}}\ Be_w, \qquad X(\theta) = \dot{\cup}_{\{w \in W \ \mid \ w \leq \theta\}}\ Be_w, \quad \theta \in W,$$

where $\leq$ denotes the Chevalley-Bruhat order. Let P_d be the maximal parabolic subgroup corresponding to the simple root α_d, and W_d be the Weyl group of P_d. We shall denote the *set of minimal reperesentatives of* W/W_{P_d} in W by W^d, namely

$$W^d = \{w \in W \mid l(ww') = l(w) + l(w'),\ \text{for all } w' \in W_d\}.$$

1.1. **Sing** $X(w)$**.** Let Sing $X(w)$ denote the singular locus of $X(w)$. If $X(w)$ is not smooth, then Sing $X(w)$ is a non-empty B-stable closed subvariety of $X(w)$. Given a point $x \in X(w)$, to decide if it is a smooth point or not, it suffices (in view of Bruhat decomposition) to determine if the T-fixed point e_τ of the B-orbit through x is a smooth point or not.

1.2. **The space** $T(w,\tau)$**.** For $\tau \leq w$, let $T(w,\tau)$ be the Zariski tangent space to $X(w)$ at e_τ. Let

$$N(w,\tau) = \{\beta \in \tau(R^+) \mid X_{-\beta} \in T(w,\tau)\}.$$

Then as remarked in the Introduction, $T(w,\tau)$ is spanned by $\{X_{-\beta},\ \beta \in N(w,\tau)\}$.
Recall the following:

Theorem 1.3. *([?], Theorem 3.2) Let $w,\tau \in W$, $w \geq \tau$. Let $\beta \in \tau(R^+)$. Then $\beta \in N(w,\tau)$ if and only if $X_{-\beta}u_\tau(\omega_d) \in V_w(\omega_d)$, for all $1 \leq d \leq l$, l being the rank of G.*

2. Bases $\mathcal{B}$ and $\mathcal{B}*$ for $V_k(\omega_d)$ and $H^0(G/B, L_{\omega_d})$.

Let G be classical. For $1 \leq d \leq l$ (l being the rank of G), let P_d, W_d, W^d be as in §1. For $w \in W^d$, we shall denote the associated Schubert variety in G/P_d by $X_{P_d}(w)$ $(= \overline{BwP_d}(\mathrm{mod}P_d))$. Let $\omega_d, 1 \leq d \leq l$ be the fundamental weights of G. Note that $(\omega_d, \beta^*) \leq 2$, for all $\beta \in R^+$.
We recall below the basic results from [**?**].

2.1. **Chevalley multiplicity.** Let $\tau, \phi \in W^d$ be such that $X_{P_d}(\phi)$ is a Schubert divisor in $X_{P_d}(\tau)$. Let $\phi = s_\beta \tau$, where $\beta \in R^+$. Let $m(\tau,\phi) = (\phi(\omega_d), \beta^*)$ (note that $(\phi(\omega_d), \beta^*) > 0$). We refer to $m(\tau,\phi)$ as the *Chevalley multiplicity of $X_{P_d}(\phi)$ in $X_{P_d}(\tau)$* .

2.2. **Admissible pairs.** A pair of elements $\tau, \phi \in W^d$, $\tau \geq \phi$ is called *an admissible pair*, if either $\tau = \phi$ (in which case we call (τ,τ) as a *trivial admissible pair*), or there exists a chain $\tau = \tau_0 > \tau_1 > \cdots > \tau_r = \phi$ such that $X_{P_d}(\tau_{i+1})$ is a divisor in $X_{P_d}(\tau_i)$, and $m(\tau_i, \tau_{i+1}) = 2$, $0 \leq i \leq r-1$.

Proposition 2.3. *Let $\tau, \phi \in W^d$ be a non-trivial admissible pair, and $\tau = \tau_0 > \tau_1 > \cdots > \tau_r = \phi$ any chain (so that $X_{P_d}(\tau_{i+1})$ is a divisor in $X_{P_d}(\tau_i)$). Let $\tau_{i+1} = s_{\beta_i}\tau_i$, $\beta_i \in R^+$, $0 \leq i \leq r-1$. Define $v \in V(\omega_d)$ as $v = X_{-\beta_0}X_{-\beta_1}\cdots X_{-\beta_{r-1}}u_\phi(\omega_d)$, $u_\phi(\omega_d)$ being as in the introduction with $\lambda = \omega_d$. Then v is independent of the chain chosen and depends only on τ and ϕ. Further, v is a weight vector of weight $\frac{1}{2}(\tau(\omega_d) + \phi(\omega_d))$.*

2.4. **The sets $\mathcal{B}$ and $\mathcal{B}_w$.** Let $\tau, \phi \in W^d$ be such that (τ,ϕ) is an admissible pair. If $\tau = \phi$, then set $q_{\tau,\tau}$ (or just q_τ) equal to $u_\tau(\omega_d)$. If $\tau > \phi$, then set $q_{\tau,\phi}$ as the vector v as given by Proposition **??**. Set $\mathcal{B} = \{q_{\tau,\phi}, (\tau,\phi) \text{ an admissible pair}\}$. For $w \in W^d$, set $\mathcal{B}_w = \{q_{\tau,\phi} \in \mathcal{B} \mid w \geq \tau\}$.

Theorem 2.5. *With notations as above, the set $\mathcal{B}$ is a basis for $V(\omega_d)$. Further, for $w \in W^d$, the set $\mathcal{B}_w$ is a basis for $V_w(\omega_d)$.*

2.6. **The sets $\mathcal{B}^*$ and $\mathcal{B}^*_w$.** Define $\mathcal{B}^*$ to be the basis of $H^0(G/P_d, L_{\omega_d})$ $(= V_K(\omega_d)^*)$ dual to $\mathcal{B}$ (here, L_{ω_d} denotes the ample generator of $\mathrm{Pic}(G/P_d)$ $(\simeq \mathbb{Z})$). Let us denote the elements of $\mathcal{B}^*$ by $\{p_{\tau,\phi}, (\tau, \phi)$ an admissible pair$\}$. For $w \in W^d$, set $\mathcal{B}^*_w = \{p_{\tau,\phi}|_{X_{P_d}(w)},\ p_{\tau,\phi} \in \mathcal{B}^* \mid p_{\tau,\phi}|_{X_{P_d}(w)} \neq 0\}$.

Theorem 2.7. *For $w \in W^d$, the set $\mathcal{B}^*_w$ is a basis of $H^0(X_{P_d}(w), L_{\omega_d})$.*

3. THE LINEAR GROUP $SL(n)$

Let $G = SL(n)$, the special linear group of rank $n-1$. Let T be the maximal torus consisting of all the diagonal matrices in G, and B the Borel subgroup consisting of all the upper triangular matrices in G. It is well-known that W can be identified with S_n, the symmetric group on n letters. Any $w \in S_n$ is usually written as $(w(1), w(2), \cdots w(n))$.

Following [**?**], we denote the simple roots by $\epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq n-1$ (note that $\epsilon_i - \epsilon_{i+1}$ is the character sending $\mathrm{diag}(t_1, \ldots, t_n)$ to $t_i t_{i+1}^{-1}$). Then $R = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n,\ i \neq j\}$, and the reflection $s_{\epsilon_i - \epsilon_j}$ may be identified with the transposition (i, j) in S_n.

3.1. **The partially ordered set $I_{d,n}$.** Fix d, $1 \leq d \leq n-1$. We have,

$$P_d = \left\{ A \in G \;\middle|\; A = \begin{pmatrix} * & * \\ 0_{(n-d)\times d} & * \end{pmatrix} \right\},\ W_d = \mathcal{S}_\lceil \times \mathcal{S}_{\backslash-\lceil}.$$

and W^d may be identified with

$$I_{d,n} := \{\underline{i} = (i_1, \ldots, i_d) \mid 1 \leq i_1 < \cdots < i_d \leq n\}.$$

Given $\underline{i}$, $\underline{j} \in I_{d,n}$, let $X_{\underline{i}}$, $X_{\underline{j}}$ be the associated Schubert varieties in G/P_d. We have

$$X_{\underline{i}} \supseteq X_{\underline{j}} \iff \underline{i} \geq \underline{j} \iff i_t \geq j_t, \text{ for all } 1 \leq t \leq d.$$

(see [**?**] for details)). In the sequel, we shall denote an element $(a_1 \ldots a_n) \in W^d$ by just $(a_1 \ldots a_d)$.

3.2. **The Chevalley-Bruhat order on S_n.** For $(a_1 \ldots a_n), (b_1 \ldots b_n) \in S_n$,

$$(a_1 \ldots a_n) \geq (b_1 \ldots b_n) \iff (a_1 \ldots a_d) \uparrow \geq (b_1 \ldots b_d) \uparrow, \text{ for all } 1 \leq d \leq n-1$$

(here, for a d-tuple $(t_1 \ldots t_d)$ of distinct integers, $(t_1 \ldots t_d) \uparrow$ denotes the ordered d-tuple obtained from $\{t_1, \ldots, t_d\}$ by arranging its elements in ascending order).

3.3. **The bases $\mathcal{B}_d$ and $\mathcal{B}_d^*$.** Let $G = SL(n)$, and $V = K^n$. We denote the standard basis for K^n by $\{e_1, \cdots, e_n\}$. Given a positive root $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$, the element $X_{-\beta}$ of the Chevalley basis of $\mathfrak{g}$ is given by $X_{-\beta} = E_{kj}$, where E_{kj} is the elementary matrix with 1 at the (k, j)-th place, and 0's elsewhere. For $1 \leq d \leq l\,(= n-1)$, we have, $V(\omega_d) = \wedge^d V$, and $q_{id} (= u(\omega_d)) = e_1 \wedge \cdots \wedge e_d$ (more generally, given $w = (a_1 \cdots a_n) \in W$, denoting by $w^{(d)}$ the element in W^d which represents the coset wW_d, we have, $q_{w^{(d)}} (= u_w(\omega_d)) = e_{a_1} \wedge \cdots \wedge e_{a_d}$ (up to ± 1)). We have,

$$X_{-\beta} e_i = \begin{cases} 0, & \text{if } i \neq j \\ e_k, & \text{if } i = j. \end{cases}$$

From this it follows that $X_{-\beta} q_{id} \neq 0$ if and only if $j \in \{1, \cdots d\}$, and $k \notin \{1, \cdots, d\}$, i.e., if and only if $j \leq d < k$; further, for $j \leq d < k$, we have,
$X_{-\beta} q_{id} = \pm e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_d \wedge e_k$. Hence we obtain

$$X_{-\beta} q_{id} = \pm q_{s_\beta^{(d)}}.$$

This implies that $\beta \in N(w, e)$ if and only if $w^{(d)} \geq s_\beta^{(d)}, j \leq d < k$, i.e., if and only if $w \geq s_\beta$ (note that for $d < j$, or $d \geq k$, $s_\beta^{(d)} = (1 \cdots d)$), where e denotes the identity element in S_n. Hence we obtain (cf.[?])

Theorem 3.4. *Let $G = SL(n)$, and $w \in S_n$. Then $T(w, e)$ is spanned by $\{X_{-\beta}, \beta \in R^+ \mid w \geq s_\beta\}$.*

Theorem 3.5. *Let $w \in S_n$, and $\beta \in R^+$. Then $\beta \in N(w, e)$ if and only if $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq l\,(= n-1)$, where $m(\omega_d - \beta)$ (resp. $m_w(\omega_d - \beta)$) denotes the multiplicity of $\omega_d - \beta$ in $V(\omega_d)$ (resp.$V_w(\omega_d)$).*

Proof. Given $d, 1 \leq d \leq l$, and $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$, from our discussions above, we see easily that

$$m(\omega_d - \beta) = \begin{cases} 0, & \text{if } d < j \text{ or } d \geq k \\ 1, & \text{if } j \leq d < k. \end{cases}$$

Hence we obtain that $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq l$ if and only if for all d, $j \leq d < k$, $w \geq s_\beta^{(d)}$, i.e., if and only if $w \geq s_\beta$. This together with Theorem **??** implies the required result. ☐

3.6. Let $w \in W$, $\beta \in R^+$. Consider the following three conditions:

(1) $w \geq s_\beta$.
(2) $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq l$, l being the rank of G.

(3) $\beta \in N(w,e)$.

Theorem 3.7. *Let G be of Type $\mathbf{A}_1$. Let $w \in W$, $\beta \in R^+$. Then the three conditions in §**??** are equivalent.*

Proof. The result follows from Theorems **??** and **??**. □

4. The Symplectic Group $Sp(2n)$.

Let $V = K^{2n}$ together with a nondegenerate, skew-symmetric bilinear form $(,)$. Let $H = SL(V)$ and $G = Sp(V) = \{A \in SL(V) \mid A$ leaves the form $(,)$ invariant $\}$. Taking the matrix of the form (with respect to the standard basis $\{e_1, ..., e_{2n}\}$ of V) to be

$$E = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$$

where J is the anti diagonal $(1, \cdots, 1)$ of size $n \times n$, we may realize $Sp(V)$ as the fixed point set of a certain involution σ on $SL(V)$, namely $G = H^\sigma$, where $\sigma : H \longrightarrow H$ is given by $\sigma(A) = E({}^tA)^{-1}E^{-1}$. We note that the following hold (cf. [**?**]):

(I) Denoting by W_G the Weyl group of G, we have

$$W_G = \{(a_1 ... a_{2n}) \in S_{2n} \mid a_i = 2n + 1 - a_{2n+1-i},\ 1 \le i \le 2n\}.$$

Thus $w = (a_1 ... a_{2n}) \in W_G$ is known once $(a_1 ... a_n)$ is known.
In the sequel, we shall denote an element $(a_1 ... a_{2n})$ in W_G by just $(a_1 ... a_n)$.

(II). We shall index the simple roots in G as in [**?**]. Let us denote the simple reflections in W_G by $\{s_i,\ 1 \le i \le n\}$. We have (cf. [**?**]),

$$s_i = \begin{cases} r_i r_{2n-i}, & \alpha_i = \varepsilon_i - \varepsilon_{i+1},\ 1 \le i \le n-1 \\ r_n, & \alpha_i = 2\varepsilon_n \end{cases}$$

where r_i denotes the transposition $(i, i+1)$ in S_{2n}, $1 \le i \le 2n-1$.

(III). For $1 \le d \le n$, W_G^d can be identified with

$\{(a_1 \cdots a_d)\, |(1)\ 1 \le a_1 < a_2 < \cdots < a_d \le 2n (2) \text{ for } 1 \le i \le 2n, \text{ if } i \in \{a_1, ..., a_d\} \text{then } 2n +$

In the sequel, we shall denote an element $(a_1 \cdots a_{2n})$ in W_G^d by just $(a_1 \cdots a_d)$.

(IV). For $w_1 = (a_1 \cdots a_n)$, $w_2 = (b_1 \cdots b_n)$, $w_1, w_2 \in W_G$, we have $w_2 \ge w_1 \Leftrightarrow$ the d-tuple $\{b_1, ..., b_d$ arranged in ascending order $\} \ge$ the d-tuple $\{a_1, ..., a_d$ arranged in ascending order$\}$, $1 \le d \le n$ (cf. [**?**]). In particular, the partial order on W_G is induced by the partial order on W_H (cf. [**?**]).

4.1. **Chevalley Basis.** For $1 \leq i \leq 2n$, set $i' = 2n+1-i$. The involution $\sigma : SL(2n) \to SL(2n), A \mapsto E({}^tA)^{-1}E^{-1}$, induces an involution $\sigma : sl(2n) \to sl(2n), A \mapsto -E({}^tA)E^{-1}(= E({}^tA)E$, since $E^{-1} = -E)$. In particular, we have, for $1 \leq i, j \leq 2n$

$$\sigma(E_{ij}) = \begin{cases} -E_{j'i'}, & \text{if } i,j \text{ are both } \leq n \text{ or both } > n \\ E_{j'i'}, & \text{if one of } \{i,j\} \text{ is } \leq n \text{ and the other } > n. \end{cases}$$

where E_{ij} is the elementary matrix with 1 at the (i,j)th place and 0 elsewhere. Further

$$\text{Lie } G = \{A \in \text{sl}(2n) \mid E({}^tA)E = A\}.$$

The Chevalley basis $\{H_{\epsilon_i - \epsilon_{i+1}}, 1 \leq i < n, H_{2\epsilon_n}, X_{\pm 2\epsilon_m}, 1 \leq m \leq n, X_{\pm(\epsilon_j - \epsilon_k)}, X_{\pm(\epsilon_j + \epsilon_k)}, 1 \leq j < k \leq n\}$ for Lie G may be given as follows:

$$H_{\epsilon_i - \epsilon_{i+1}} = E_{ii} - E_{i+1,i+1} + E_{(i+1)',(i+1)'} - E_{i'i'},\ H_{2\epsilon_n} = E_{nn} - E_{n'n'},$$
$$X_{\epsilon_j - \epsilon_k} = E_{jk} - E_{k'j'},\ X_{\epsilon_j + \epsilon_k} = E_{jk'} + E_{kj'},\ X_{2\epsilon_m} = E_{mm'},$$
$$X_{-(\epsilon_j - \epsilon_k)} = E_{kj} - E_{j'k'},\ X_{-(\epsilon_j + \epsilon_k)} = E_{k'j} + E_{j'k},\ X_{-2\epsilon_m} = E_{m'm}.$$

Definition 4.2. Let $\phi = (a_1 \cdots a_d) \in W^d$, and let $i,\ 1 \leq i \leq n$ be such that $i, (i+1)' \in \{a_1, \cdots, a_d\}$. Let $\tau = (b_1 \cdots b_d)$ be the element of W^d obtained from $(a_1 \cdots a_d)$ by replacing i by $i+1$, and $(i+1)'$ by i'. In this situation, we say that τ is obtained from ϕ by a *Type I* operation.

Proposition 4.3. *(cf. [?]) Let $\tau, \phi \in W^d, \tau \geq \phi$. Then (τ, ϕ) is an admissible pair if and only if either $\tau = \phi$, or τ is obtained from ϕ by a sequence of Type I operations.*

4.4. **The G-module $V(\omega_d)$.** For $1 \leq d \leq n$, we have $\omega_d = \epsilon_1 + \cdots + \epsilon_d$, where $\{\epsilon_1, \cdots, \epsilon_{2n}\}$ is the canonical basis of Hom $(D_{2n}, \mathbf{G_m})$ (D_{2n} being the maximal torus in GL $(2n)$ consisting of all the diagonal matrices). If $d = 1$, then $V(\omega_d) = V(= K^{2n})$. Let us then suppose that $d \geq 2$. Consider the 2-form $f \in \wedge^2 V$ given by

$$f = e_1 \wedge e_{2n} + e_2 \wedge e_{2n-1} + \cdots + e_n \wedge e_{n+1}$$

(here $\{e_1, \cdots, e_{2n}\}$ is the standard basis in V). We have

$$V(\omega_d) = \{\text{ the primitive vectors in } \wedge^d V\}$$
$$= \{v \in \wedge^d V \mid v \wedge f^{n+1-d} = 0\}.$$

The extremal weight vectors $\{q_\tau, \tau \in W^d\}$, say $\tau = (a_1 \cdots a_d)$ are given by

$$q_\tau = e_{a_1} \wedge \cdots \wedge e_{a_d}$$

.

Proposition 4.5. *(cf. [?]) Let $\beta \in R^+$.*

(1) *Let* $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$. *Then*

$$X_{-\beta}q_{id} = \begin{cases} 0, & \text{if } d < j \text{ or } d \geq k \\ \pm q_{s_\beta^{(d)}}, & \text{if } j \leq d < k. \end{cases}$$

(2) *Let* $\beta = 2\epsilon_j, 1 \leq j \leq n$. *Then*

$$X_{-\beta}q_{id} = \begin{cases} 0, & \text{if } d < j \\ \pm q_{s_\beta^{(d)}}, & \text{if } j \leq d \leq n. \end{cases}$$

(3) *Let* $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$. *Then*

$$X_{-\beta}q_{id} = \begin{cases} 0, & \text{if } d < j \\ \pm q_{s_\beta^{(d)}}, & \text{if } j \leq d < k \\ \pm q_{\tau,\phi}, & \text{if } k \leq d \leq n \end{cases}$$

where $\tau = (12 \cdots j-1\, j+1 \cdots dj')$ *and* $\phi = (12 \cdots k-1\, k+1 \cdots dk')$.

Theorem 4.6. *Let* $w \in W$, *and* $\beta \in R^+$.

(1) *Let* $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$, *or* $2\epsilon_j, 1 \leq j \leq n$. *Then* $\beta \in N(w, e)$ *if and only if* $w \geq s_\beta$.

(2) *Let* $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$. *Then* $\beta \in N(w, e)$ *if and only if* $w \geq$ *either* s_β *or* $s_{2\epsilon_j}$.

Proof. If $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$, or $2\epsilon_j, 1 \leq j \leq n$, then the result is immediate from (1) and (2) of Proposition **??** (in view of Theorem **??**).

Let then $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$. We have (from (3) of Proposition **??**),

$$X_{-\beta}q_{id} = \pm q_{s_\beta^{(d)}},\ j \leq d < k. \tag{*}$$

For $k \leq d \leq n$, we have $X_{-\beta}q_{id} = \pm q_{\tau,\phi}$, τ, ϕ being as in Proposition **??**,(3). We have $s_{2\epsilon_j} = (12 \cdots j-1\, j'\, j+1 \cdots n)$, and hence we obtain

$$X_{-\beta}q_{id} = \pm q_{s_{2\epsilon_j}^{(d)},\phi},\ k \leq d \leq n \tag{**}$$

Hence from (*) and (**) we obtain (in view of Theorem **??**) that $\beta \in N(w, e)$ if and only if $w^{(d)} \geq s_\beta^{(d)}, j \leq d < k$ and $w^{(d)} \geq s_{2\epsilon_j}^{(d)}, k \leq d \leq n$.

Claim. $w^{(d)} \geq s_\beta^{(d)}, j \leq d < k$ and $w^{(d)} \geq s_{2\epsilon_j}^{(d)}, k \leq d \leq n$ if and only if $w \geq$ either s_β or $s_{2\epsilon_j}$.

If $w \geq$ either s_β or $s_{2\epsilon_j}$, then clearly $w^{(d)} \geq s_\beta^{(d)}, j \leq d < k$ and $w^{(d)} \geq s_{2\epsilon_j}^{(d)}, k \leq d \leq n$ (note that $s_\beta^{(d)} > s_{2\epsilon_j}^{(d)}, k \leq d \leq n$, and $s_{2\epsilon_j}^{(d)} > s_\beta^{(d)}, j \leq d < k$).

Let now w be such that $w^{(d)} \geq s_\beta^{(d)}, j \leq d < k$ and $w^{(d)} \geq s_{2\epsilon_j}^{(d)}, k \leq d \leq n$. Further, let $w \not\geq s_\beta$. We shall now show that $w \geq s_{2\epsilon_j}$. Let $w = (a_1 \cdots a_n)$. The facts that $w^{(d)} \geq s_\beta^{(d)}, j \leq d < k$, and $w \not\geq s_\beta$ imply that

$$w^{(k)} \not\geq s_\beta^{(k)}. \qquad (\dagger)$$

On the other hand, we have $w^{(k)} \geq s_{2\epsilon_j}^{(k)}$ (since $w^{(d)} \geq s_{2\epsilon_j}^{(d)}, k \leq d \leq n$). This implies that there is an entry, say x, in $\{a_1, \cdots, a_k\}$ such that $x \geq j'$. In fact, this entry x belongs to $\{a_1, \cdots, a_j\}$; for, if $x \notin \{a_1, \cdots, a_j\}$, then this would imply that there is an entry say y, in $\{a_1, \cdots, a_j\}$ such that $y \geq k'$ (since $w^{(j)} \geq s_\beta^{(j)}$), which in turn would imply that $w^{(k)} \geq s_\beta^{(k)}$ (since $x, y \in \{a_1, \cdots, a_j\}$, and $x \geq j', y \geq k'$) and this is not true (cf. $(\dagger)$). It follows that $w \geq s_{2\epsilon_j}$. This completes the proof of the Claim and also of Theorem **??**. □

Proposition 4.7. *Let $\beta \in R^+$.*

(1) *Let $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$. Then*

$$m(\omega_d - \beta) = \begin{cases} 0, & \text{if } d < j, \text{ or } d \geq k \\ 1, & \text{if } j \leq d < k. \end{cases}$$

(2) *Let $\beta = 2\epsilon_j, 1 \leq j \leq n$. Then*

$$m(\omega_d - \beta) = \begin{cases} 0, & \text{if } d < j \\ 1, & \text{if } j \leq d \leq n. \end{cases}$$

(3) *Let $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$. Then*

$$m(\omega_d - \beta) = \begin{cases} 0, & \text{if } d < j \\ 1, & \text{if } j \leq d < k \\ n+1-d, & \text{if } k \leq d \leq n. \end{cases}$$

Proof. Fix $d, 1 \leq d \leq n$.
(1) Let $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$. If $d < j$ or $d \geq k$, then clearly $m(\omega_d - \beta) = 0$. Let then $j \leq d < k$. Then $q_{s_\beta^{(d)}}$ is the only vector in $\mathcal{B}_d$ of weight $\omega_d - \beta$.
(2) Let $\beta = 2\epsilon_j, 1 \leq j \leq n$. If $d < j$, then clearly $m(\omega_d - \beta) = 0$. Let then $j \leq d \leq n$. Then $q_{s_\beta^{(d)}}$ is the only vector in $\mathcal{B}_d$ of weight $\omega_d - \beta$.
(3) Let $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$. If $d < j$, then clearly $m(\omega_d - \beta) = 0$. Let then $j \leq d \leq n$. If $d < k$, then clearly $q_{s_\beta^{(d)}}$ is the only vector in $\mathcal{B}_d$ of weight $\omega_d - \beta$. If $k \leq d \leq n$, then using the description of admissible pairs (cf. Proposition **??**) and the fact that $q_{\theta,\delta}$ is a weight vector of

weight $\frac{1}{2}(\theta(\omega_d) + \delta(\omega_d))$, it is easily checked that there are precisely $n+1-d$ vectors in $\mathcal{B}_d$ of weight $\omega_d - \beta$, namely
$q_{\tau,\phi},\ q_{\theta_i,\delta_i},\ 0 \leq i \leq n-d-1$ where
$\tau = (1 \cdots j-1\, j+1 \cdots dj'), \phi = (1 \cdots k-1\, k+1 \cdots dk')$,
$\theta_0 = (1 \cdots j-1\, j+1 \cdots k-1\, k+1 \cdots d\, d+1\, k')$,
$\delta_0 = (1 \cdots j-1\, j+1 \cdots d(d+1)')$,
$\theta_i = (12 \cdots j-1\, j+1 \cdots k-1\, k+1 \cdots (d+i+1)(d+i)')$,
$\delta_i = (12 \cdots j-1\, j+1\, \cdots k-1\, k+1 \cdots (d+i)(d+i+1)'),\ 1 \leq i \leq n-d-1$.
□

Corollary 4.8. *Let $w \in W$ and $\beta \in R^+$.*

(1) *Let $\beta = \epsilon_j - \epsilon_k,\ j < k \leq n,\ 2\epsilon_m,\ 1 \leq m \leq n$. Then $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$ if and only if $w \geq s_\beta$.*
(2) *Let $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$. Then $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$ if and only if $w \geq s_\beta$ or $s_{2\epsilon_j} s_{\epsilon_k - \epsilon_n}$ $(= (12 \cdots j-1 j' j+1 \cdots k-1 n k+1 k+2 \cdots n-1 k))$.*

Proof. (1) If $\beta = \epsilon_j - \epsilon_k,\ j < k \leq n,\ 2\epsilon_m,\ 1 \leq m \leq n$, then the result follows from (1), (2) in the proof of Proposition 4.7.
(2) Let $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$. We have that if $d < j$, then $m(\omega_d - \beta) = 0$, and if $j \leq d < k$, then $m(\omega_d - \beta) = 1$ ($q_{id^{(d)}}$ being the only weight vector in $V(\omega_d)$ of weight $\omega_d - \beta$). For $k \leq d \leq n$, we have from the proof of (3) in Proposition **??** that the weight space in $V(\omega_d)$ of weight $\omega_d - \beta$ has a basis consisting of the vectors $q_{\tau,\phi},\ q_{\theta_i,\delta_i},\ 0 \leq i \leq n-d-1$ (notations being as in the proof of (3) in Proposition **??**). Hence we obtain that $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$ if and only if $w^{(j)} \geq \{1, 2, \cdots, j-1, k'\}$ and $w^{(d)} \geq \tau$ and $\theta_i,\ 0 \leq i \leq n-d-1,\ k \leq d \leq n$. It is now easily checked $w^{(d)} \geq q_{\tau,\phi},\ q_{\theta_i,\delta_i},\ 0 \leq i \leq n-d-1,\ k \leq d \leq n$, if and only if $w^{(k)} \geq \{1, 2 \cdots j-1 j' j+1 \cdots k-1 n\}$. The required result now follows from this. □

Remark 4.9. Let $w \in W$, and $\beta \in R^+$.
The condition that $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$ need not be equivalent to the condition that $\beta \in N(w, e)$. For example, take $w = s_{2\epsilon_j}$ for some $j < n-1$, $\beta = \epsilon_j + \epsilon_k$ for some $k,\ j < k \leq n-1$. We have (cf. Theorem **??**) $\beta \in N(w, e)$, but $m_w(\omega_d - \beta) \neq m(\omega_d - \beta),\ k \leq d < n$ (note that $m_w(\omega_d - \beta) = 1,\ k \leq d \leq n$, while $m(\omega_d - \beta) = n+1-d,\ k \leq d \leq n$).
Also, the condition that $w \geq s_\beta$ need not be equivalent to the condition that $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$. For example, take $\beta = \epsilon_j + \epsilon_k$, for some $j < k \leq n$, and $w = s_{2\epsilon_j} s_{\epsilon_k - \epsilon_n}$. We have, $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$ (cf. Corollary **??**), but $w \not\geq s_\beta$.

Of course, for $\beta = \epsilon_j - \epsilon_k,\ j < k \leq n,\ 2\epsilon_m,\ 1 \leq m \leq n$, and any $w \in W$, all the three conditions of §**??** are equivalent.

5. The Orthogonal Group SO$(2n+1)$

Let $V = K^{2n+1}$ together with a non degenerate symmetric bilinear form (,). Taking the matrix of the form (,) (with respect to the standard basis $\{e_1, \cdots, e_{2n+1}\}$ of V) to be E, the $2n+1 \times 2n+1$ anti-diagonal matrix with 1 all along the anti-diagonal except at the $n+1 \times n+1$-th place where the entry is 2 (note that the associated quadratic form Q on V is given by $Q(\sum_{i=1}^{2n+1} x_i e_i) = x_{n+1}^2 + \sum_{i=1}^{n} x_i x_{2n+2-i}$), we may realize $G = SO(V)$ as the fixed point set $SL(V)^\sigma$, where $\sigma : SL(V) \to SL(V)$ is given by $\sigma(A) = E^{-1}({}^tA)^{-1}E$. Set $H = SL(V)$.

We note that the following hold (cf. [**?**]):

(I). Denoting by W_G the Weyl group of G, we have

$$W_G = \{(a_1 ... a_{2n+1}) \in S_{2n+1} \mid a_i = 2n+2 - a_{2n+2-i},\ 1 \leq i \leq 2n+1\}.$$

Thus $w = (a_1 ... a_{2n+1}) \in W_G$ is known once $(a_1 ... a_n)$ is known (note that $a_{n+1} = n+1$, for all $w \in W_G$).
In the sequel, we shall denote an element $(a_1 ... a_{2n+1})$ in W_G by just $(a_1 ... a_n)$.

(II). We shall index the simple roots in G as in [**?**]. Let us denote the simple reflections in W_G by $\{s_i,\ 1 \leq i \leq n\}$. We have (cf. [**?**]),

$$s_i = \begin{cases} r_i r_{2n+1-i}, & \alpha_i = \varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq n-1, \\ r_n r_{n+1} r_n, & \alpha_i = \varepsilon_n \end{cases}$$

where r_i denotes the transposition $(i, i+1)$ in $S_{2n+1},\ 1 \leq i \leq 2n$.

(III). For $1 \leq d \leq n$, W_G^d can be identified with

$$\{(a_1 \cdots a_d) \,|(1)\ 1 \leq a_1 < a_2 < \cdots < a_d \leq 2n+1,\ a_i \neq n+1, 1 \leq i \leq d(2) \text{ for } 1 \leq i \leq 2n+$$

In the sequel, we shall denote an element $(a_1 \cdots a_{2n+1})$ in W_G^d by just $(a_1 \cdots a_d)$.

(IV). For $w_1 = (a_1 \cdots a_n),\ w_2 = (b_1 \cdots b_n),\ w_1, w_2 \in W_G$, we have $w_2 \geq w_1 \Leftrightarrow$ the d-tuple $\{b_1, ..., b_d$ arranged in ascending order $\} \geq$ the d-tuple $\{a_1, ..., a_d$ arranged in ascending order$\}$, $1 \leq d \leq n$ (cf. [**?**]). In particular, the partial order on W_G is induced by the partial order on W_H.

5.1. **Chevalley Basis.** For $1 \leq k \leq 2n+1$, set $k' = 2n+2-k$. The involution $\sigma : SL(2n+1) \to SL(2n+1), A \mapsto E^{-1}({}^tA)^{-1}E$, induces an involution $\sigma : sl(2n+1) \to sl(2n+1), A \mapsto -E^{-1}({}^tA)E$. In particular, we have, $\sigma(E_{ij}) = -E_{j'i'}$, $1 \leq i, j \leq 2n+1$, where E_{ij} is the elementary matrix with 1 at the (i, j)th place and 0 elsewhere. Further

$$\text{Lie } G = \{A \in \text{sl}(2n+1) \mid E^{-1}({}^tA)E = -A\}.$$

The Chevalley basis $\{H_{\epsilon_i - \epsilon_{i+1}}, 1 \leq i < n, H_{\epsilon_n}, X_{\pm\epsilon_m}, 1 \leq m \leq n, X_{\pm(\epsilon_j - \epsilon_k)}, X_{\pm(\epsilon_j + \epsilon_k)}, 1 \leq j < k \leq n\}$ for Lie G may be given as follows:

$$H_{\epsilon_i - \epsilon_{i+1}} = E_{ii} - E_{i+1,i+1} + E_{(i+1)',(i+1)'} - E_{i'i'},\ H_{\epsilon_n} = 2(E_{nn} - E_{n'n'}),$$

$$X_{\epsilon_j - \epsilon_k} = E_{jk} - E_{k'j'},\ X_{\epsilon_j + \epsilon_k} = E_{jk'} - E_{kj'},\ X_{\epsilon_m} = 2E_{mn+1} - E_{n+1m'},$$

$$X_{-(\epsilon_j - \epsilon_k)} = E_{kj} - E_{j'k'},\ X_{-(\epsilon_j + \epsilon_k)} = E_{k'j} - E_{j'k},\ X_{-\epsilon_m} = 2E_{n+1m} - E_{m'n+1}.$$

Definition 5.2. Let $\phi = (a_1 \cdots a_d) \in W^d$, $1 \leq d \leq n-1$, and let $n \in \{a_1, \cdots, a_d\}$. Let $\tau = (b_1 \cdots b_d)$ be the element of W^d obtained from $(a_1 \cdots a_d)$ by replacing n by n'. In this situation, we say that τ is obtained from ϕ by a *Type II* operation.

Remark 5.3. *Type I* operation is defined exactly as in Definition **??**

Proposition 5.4. *(cf. [*?*]) Let $\tau, \phi \in W^d$, $1 \leq d \leq n-1, \tau \geq \phi$. Then (τ, ϕ) is an admissible pair if and only if either $\tau = \phi$, or τ is obtained from ϕ by a sequence of operations of Type I or II.*

5.5. **The G-module $V(\omega_d)$.** For $d = n$, $V(\omega_d)$ is the spin representation, and the extremal weight vectors, $q_\tau, \tau \in W^d$ form a basis for $V(\omega_d)$. For $1 \leq d < n$, we have $V(\omega_d) = \wedge^d V$ (here, $V = K^{2n+1}$). The extremal weight vectors $\{q_\tau . \tau \in W^d\}$, say $\tau = (a_1 \cdots a_d)$ are given by

$$q_\tau = e_{a_1} \wedge \cdots \wedge e_{a_d}$$

.

Proposition 5.6. *(cf. [*?*]) Let $\beta \in R^+$.*

(1) *Let $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$. Then*

$$X_{-\beta} q_{id} = \begin{cases} 0, & \text{if } d < j \text{ or } d \geq k \\ \pm q_{s_\beta^{(d)}}, & \text{if } j \leq d < k. \end{cases}$$

(2) *Let $\beta = \epsilon_j, 1 \leq j \leq n$.*

(a) *If $j = n$, then*

$$X_{-\beta} q_{id} = \begin{cases} 0, & \text{if } d < n \\ \pm q_{s_\beta^{(d)}}, & \text{if } d = n. \end{cases}$$

(b) *If $j < n$, then*

$$X_{-\beta}q_{id} = \begin{cases} 0, & \text{if } d < j \\ \pm q_{s_\beta^{(d)}}, & \text{if } d = n \\ \pm q_{s^{(d)}_{\epsilon_j+\epsilon_n}, s^{(d)}_{\epsilon_j-\epsilon_n}}, & \text{if } j \le d < n. \end{cases}$$

(3) *Let $\beta = \epsilon_j + \epsilon_k, 1 \le j < k \le n$.*
(a) *If $k = n$, then*

$$X_{-\beta}q_{id} = \begin{cases} 0, & \text{if } d < j \\ \pm q_{s_\beta^{(d)}}, & \text{if } j \le d. \end{cases}$$

(b) *If $k < n$, then*

$$X_{-\beta}q_{id} = \begin{cases} 0, & \text{if } d < j \\ \pm q_{s_\beta^{(d)}}, & \text{if } j \le d < k \text{ or } d = n \\ \pm(\sum_{i=0}^{n-d} c_i q_{\theta_i,\delta_i} + a q_{\tau,\phi}), & \text{if } k \le d < n \end{cases}$$

where $c_i = \pm 2,\ i < n-d,\ c_{n-d} = \pm 1 = a$,
$\tau = (1\cdots j-1\, j+1\cdots dj'), \phi = (1\cdots k-1\, k+1\cdots dk')$,
$\theta_0 = (1\cdots j-1\, j+1\cdots k-1\, k+1\cdots d\, d+1\, k')$,
$\delta_0 = (1\cdots j-1\, j+1\cdots d(d+1)')$,
$\theta_i = (12\cdots j-1\, j+1\cdots k-1\, k+1\cdots (d+i+1)(d+i)')$,
$\delta_i = (12\cdots j-1\, j+1\, \cdots k-1\, k+1\cdots (d+i)(d+i+1)'),\ 1 \le i < n-d$,
$\theta_{n-d} = (12\cdots j-1\, j+1\cdots k-1\, k+1\cdots n'(n-1)')$,
$\delta_{n-d} = (12\cdots j-1\, j+1\cdots k-1\, k+1\cdots n-1n)$.

Theorem 5.7. *Let $w \in W$, and $\beta \in R^+$.*

1. *Let $\beta = \epsilon_j - \epsilon_k, 1 \le j < k \le n,\ \epsilon_n$, or $\epsilon_i + \epsilon_n, 1 \le i \le n$. Then $\beta \in N(w,e)$ if and only if $w \ge s_\beta$.*
2. *Let $\beta = \epsilon_j, j < n$. Then $\beta \in N(w,e)$ if and only if $w \ge$ either s_β or $s_{\epsilon_j+\epsilon_n}$.*
3. *Let $\beta = \epsilon_j + \epsilon_k, 1 \le j < k < n$. Then $\beta \in N(w,e)$ if and only if $w \ge$ either s_β or $s_{\epsilon_j}s_{\epsilon_k+\epsilon_n}$.*

Proof. If $\beta = \epsilon_j - \epsilon_k, 1 \le j < k \le n,\ \epsilon_n$, or $\epsilon_i + \epsilon_n, 1 \le i \le n$, then the result follows from 1, 2(a), and 3(a) of Proposition **??** (in view of Theorem **??**).

Let $\beta = \epsilon_j, j < n$. If $w \ge$ either s_β or $s_{\epsilon_j+\epsilon_n}$, then clearly, $X_{-\beta}q_{id} \in V_w(\omega_d)$, for all $1 \le d \le n$, and hence $\beta \in N(w,e)$ (in view of Theorem **??**). Let then w be such that $X_{-\beta}q_{id} \in V_w(\omega_d)$, for all $1 \le d \le n$, Further, let $w \not\ge s_\beta$. We shall now show that $w \ge s_{\epsilon_j+\epsilon_n}$. Let $w = (a_1\cdots a_n)$. Now the fact that $w \not\ge s_\beta$ implies that $w^{(j)} \not\ge s_\beta^{(j)}$. On the other hand, we have $w^{(j)} \geq s^{(j)}_{\epsilon_j+\epsilon_n}$. Hence we obtain that there exists an entry, say x in $\{a_1 \cdots a_j\}$ such that $n' \leq x < j'$. Also, the fact that $w^{(n)} \geq s^{(n)}_\beta$ implies that there exists an entry, say y in $\{a_1 \cdots a_n\}$ such that $y \geq j'$. Denoting $\eta = s_{\epsilon_j+\epsilon_n}$, we obtain

$$w^{(j)} \geq (12 \cdots j-1\, j+1\, n')\ (= \eta^{(j)}),$$

$$w^{(n)} \geq (12 \cdots j-1\, j+1 \cdots n-1\, n'\, j')\ (= \eta^{(n)}).$$

From this it follows that $w \geq s_{\epsilon_j+\epsilon_n}$.

Let $\beta = \epsilon_j + \epsilon_k,\ 1 \leq j < k < n$. If $w \geq$ either s_β or $s_{\epsilon_j} s_{\epsilon_k+\epsilon_n}$, then clearly $X_{-\beta} q_{id} \in V_w(\omega_d)$, for all d, and hence $\beta \in N(w,e)$. Let now $\beta \in N(w,e)$, and let $w \not\geq s_\beta$. We shall now show that $w \geq s_{\epsilon_j} s_{\epsilon_k+\epsilon_n}$. We have, $w^{(d)} \geq s^{(d)}_\beta, j \leq d < k$. This implies in particular that $w^{(j)} \geq s^{(j)}_\beta$. Hence there exists an entry p in $\{a_1 \cdots a_j\}$ such that $p \geq k'$. Also, the facts that $w \not\geq s_\beta,\ w^{(j)} \geq s^{(j)}_\beta$ imply that

$$w^{(k)} \not\geq s^{(k)}_\beta. \qquad (\dagger)$$

On the other hand, since $\beta \in N(w,e)$, we have, $X_{-\beta} q_{id} \in V_w(\omega_d),\ 1 \leq d \leq n$ (cf. Theorem **??**), and hence we obtain, $w^{(k)} \geq (12 \cdots j-1\, j+1 \cdots j')\,(=\tau)$ (cf. Proposition **??**, 3(b)). Hence there exists an entry q in $\{a_1 \cdots a_k\}$ such that $q \geq j'$. We have in fact $q \in \{a_1 \cdots a_j\}$, and $q = p$ (otherwise, we would obtain $w^{(k)} \geq s^{(k)}_\beta$ contradicting ($\dagger$)).Further, we have, there exists an entry r in $\{a_1 \cdots a_k\}$ such that $r \geq n'$ (in view of Proposition **??**, 3(b)), and $r < k'$ (since $w^{(k)} \not\geq s^{(k)}_\beta$). Also, the fact that $w^{(n)} \geq s^{(n)}_\beta$ implies that there exists an entry s in $\{a_1 \cdots a_n\}$ such that $s \geq k'$. Thus we obtain, denoting $\xi = s_{\epsilon_j} s_{\epsilon_k+\epsilon_n}$,

$$w^{(j)} \geq (12 \cdots j-1\, j+1\, j')\ (= \xi^{(j)}),$$

$$w^{(k)} \geq (12 \cdots j-1\, j+1 \cdots k\, n'\, j')\ (= \xi^{(k)}),$$

$$w^{(n)} \geq (12 \cdots j-1\, j+1 \cdots k-1\, k+1 \cdots n-1\, n'\, k'\, j')\ (= \xi^{(n)}).$$

From this it follows that $w \geq s_{\epsilon_j} s_{\epsilon_k+\epsilon_n}$. □

Proposition 5.8. *Let $\beta \in R^+$. Let $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k < n$, and $k \leq d < n$. Then $m(\omega_d - \beta) = n - d + 2$. In all other cases, we have, $m(\omega_d - \beta) = 0$ or 1*

Proof. Fix $d, 1 \leq d \leq n$.
(1)Let $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$. If $d < j$ or $d \geq k$, then clearly $m(\omega_d - \beta) = 0$. Let then $j \leq d < k$. Then $q_{s^{(d)}_\beta}$ is the only vector in $\mathcal{B}_d$ of weight $\omega_d - \beta$.

(2) Let $\beta = \epsilon_j, 1 \leq j \leq n$. If $d < j$, then clearly $m(\omega_d - \beta) = 0$. Let then $j \leq d \leq n$. If $d = n$, then $q_{s_\beta^{(d)}}$ is the only vector in $\mathcal{B}_d$ of weight $\omega_d - \beta$. If $j \leq d < n$, then using the description of admissible pairs (cf. Proposition **??**) and the fact that $q_{\theta,\delta}$ is a weight vector of weight $\frac{1}{2}(\theta(\omega_d) + \delta(\omega_d))$, it is easily seen that $q_{s^{(d)}_{\epsilon_j+\epsilon_n}, s^{(d)}_{\epsilon_j-\epsilon_n}}$ is the only vector in $\mathcal{B}_d$ of weight $\omega_d - \beta$.
(3) Let $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$. If $d < j$, then clearly $m(\omega_d - \beta) = 0$. Let then $j \leq d \leq n$. If $d = n$, or $k = n$, or $j \leq d < k$, then $q_{s_\beta^{(d)}}$ is the only vector in $\mathcal{B}_d$ of weight $\omega_d - \beta$. Let then $k \leq d < n$. Then it is easily checked as in (2) (by weight considerations) that q_{θ_i,δ_i}, $0 \leq i \leq n - d$, and $q_{\tau,\phi}$ (as in Proposition **??**, 3(b)) are the only vectors in $\mathcal{B}_d$ of weight $\omega_d - \beta$.
The required result follows from (1), (2) and (3). □

Theorem 5.9. *Let $w \in W$, and $\beta \in R^+$. Then $\beta \in N(w, e)$ if and only if $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$.*

Proof. The required result follows from Proposition 5.6, Theorem **??**, and (1), (2) and (3) in the proof of Proposition **??**. □

Remark 5.10. Let $\beta \in R^+$, $w \in W$.
If $\beta = \epsilon_j - \epsilon_k$, $j < k \leq n$, ϵ_n, or $\epsilon_m + \epsilon_n$, $1 \leq m \leq n - 1$, then the three conditions in §**??** are equivalent.
For all $\beta \in R^+$, $w \in W$, the condition that $\beta \in N(w, e)$ is equivalent to the condition $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$ (cf. Theorem **??**).
The condition that $w \geq s_\beta$ need not be equivalent to the condition that $\beta \in N(w, e)$. For example, take $\beta = \epsilon_j$ for some $j < n$ and $w = s_{\epsilon_j+\epsilon_n}$; we have (cf. Theorem **??**, (2)), $\beta \in N(w, e)$, but $w \not\geq s_\beta$.

6. The Orthogonal Group SO($2n$)

Let $V = K^{2n}$ together with a non-degenerate symmetric bilinear form (,). Taking the matrix of the form (,) (with respect to the standard basis $\{e_1, \cdots, e_{2n}\}$ of V) to be E, the anti-diagonal $(1, \cdots, 1)$ of size $2n \times 2n$, we may realize $G = SO(V)$ as the fixed point set $SL(V)^\sigma$, where $\sigma : SL(V) \to SL(V)$ is given by $\sigma(A) = E({}^tA)^{-1}E$. Set $H = SL(V)$.

We note that the following hold (cf. [?]):

(I). Denoting by W_G the Weyl group of G, we have

$$W_G = \{(a_1 \cdots a_{2n}) \in S_{2n} \,|(1)\; a_i = 2n + 1 - a_{2n+1-i}, 1 \leq i \leq 2n (2)\; \#\{i, 1 \leq i \leq n\} \text{ is even}\}.$$

Thus $w = (a_1 ... a_{2n}) \in W_G$ is known once $(a_1 ... a_n)$ is known.

In the sequel, we shall denote an element $(a_1 \cdots a_{2n})$ in W by just $(a_1 \cdots a_n)$.

(II). We shall index the simple roots in G as in [?]. Let us denote the simple reflections in W_G by $\{s_i,\ 1 \leq i \leq n\}$. We have (cf. [?]),

$$s_i = \begin{cases} r_i r_{2n-i}, & \alpha_i = \varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq n-1, \\ r_n r_{n-1} r_{n+1} r_n, & \alpha_i = \varepsilon_{n-1} + \varepsilon_n. \end{cases}$$

where r_i denotes the transposition $(i, i+1)$ in $S_{2n},\ 1 \leq i \leq 2n-1$.

(III). For $1 \leq d \leq n,\ d \neq n-1$, W_G^d can be identified with
(∗).
$\{(a_1 \cdots a_d)\,|(1) 1 \leq a_1 < a_2 < \cdots < a_d \leq 2n (2) \text{for } 1 \leq i \leq 2n,\ \text{ if } i \in \{a_1, ..., a_d\} \text{then } 2n+1$

For $d = n-1$, W_G^d gets identified with a certain *proper* subset of (*); in particular, for $w_1 = (a_1 \cdots a_{2n}),\ w_2 = (b_1 \cdots b_{2n}),\ w_1, w_2 \in W_G$, we can have $w_1^{(n-1)} = w_2^{(n-1)}$, with$\{a_1, \cdots, a_{n-1}\}\uparrow,\ \{b_1, \cdots, b_{n-1}\}\uparrow$ being different. For $w \in W$, say $w = (a_1 \cdots a_{2n})$, we see easily that

$$w^{(d)} = \{a_1, \cdots, a_d\}\uparrow,\ 1 \leq d \leq n,\ d \neq n-1$$

and

$$w^{(n-1)} = \text{the least (under } \geq) \text{ in the totally ordered set } Y$$

where

$$Y = \{(y_1^{(i)}, \cdots, y_{n-1}^{(i)})\uparrow\ \ 0 \leq i \leq n,\ i \neq n-1\}.$$

$y_1^{(i)}, \cdots, y_{n-1}^{(i)}$ being the first $(n-1)$ entries in $wu_i,\ 0 \leq i \leq n,\ i \neq n-1$. (Here, the partial order $\geq$ is the usual partial order, namely, $(i_1, \cdots, i_{n-1}) \geq (j_1, \cdots, j_{n-1})$, if $i_t \geq j_t,\ 1 \leq t \leq n-1,\ (i_1, \cdots, i_{n-1}),\ (j_1, \cdots, j_{n-1})$ being two increasing sequence of $(n-1)$-tuples.)

(IV). For $1 \leq i \leq 2n$, let $i' = 2n+1-i$, and $|i| = \min\{i, i'\}$. We shall denote the Bruhat order on $W(G)$ by $\succeq$. Given $w_1 = (a_1 \cdots a_n),\ w_2 = (b_1 \cdots b_n),\ w_1, w_2 \in W_G$, we have $w_2 \succeq w_1$ if and only if the following two conditions hold (cf. [?]).

1. (1) For $1 \leq d \leq n$, we have $\{b_1, \cdots, b_d\} \uparrow \geq \{a_1, \cdots, a_d\} \uparrow$, for all d.
2. (2) Let $\{c_1, \cdots, c_d\}$ (resp. $\{e_1, \cdots, e_d\}$) be the set $\{a_1, \cdots, a_d\} \uparrow$ (resp. $\{b_1, \cdots, b_d\} \uparrow$). Suppose for some $r,\ 1 \leq r \leq d$, and some $i,\ 0 \leq i \leq d-r$, $\{|c_{i+1}|, \cdots |c_{i+r}|\} = \{|e_{i+1}|, \cdots |e_{i+r}|\} = \{n+1-r, \cdots, n\}$ (in some order). Then $\#\{j, i+1 \leq j \leq i+r \mid c_j > n\}$, and $\#\{j, i+1 \leq j \leq i+r \mid e_j > n\}$ should both be even or both odd.

Thus the Bruhat order $\succeq$ on W_G is *not* induced from the Bruhat order on W_H. Following the terminology in [?], we shall refer to condition (2) above as
" if $\{c_1, \cdots, c_d\}$ and $\{e_1, \cdots, e_d\}$ have analogous parts, then they are **D**-compatible"; we shall refer to $\{|c_{i+1}|, \cdots |c_{i+r}|\}$ and $\{|e_{i+1}|, \cdots |e_{i+r}|\}$ as analogous parts.

In the sequel, we shall have occsion to use both of the partial orders $\succeq$ and $\geq$.

Remark 6.1. (a) Let $(c_1, \cdots, c_d)$, $(e_1, \cdots, e_d) \in W_G^{P_d}$, where $(c_1, \cdots, c_d) \succeq \{e_1, \cdots, e_d)$. Suppose $(c_1, \cdots, c_d)$, $(e_1, \cdots, e_d)$ have analogous parts. Then it is easily seen that the condition (2) is equivalent to the condition that $\#\{j, 1 \leq j \leq d \mid c_j > n\}$ and $\#\{j, 1 \leq j \leq d \mid e_j > n\}$ are both even or both odd.
(b). Given $\theta \in W$, say $\theta = (a_1 \cdots a_{2n})$, denoting by $y_1^{(i)}, \cdots, y_{n-1}^{(i)}$ the first $(n-1)$ entries in θu_i, $0 \leq i \leq n$, $i \neq n-1$, we have

$$(y_1^{(i)}, \cdots, y_{n-1}^{(i)}) = \begin{cases} (x_1, \cdots, x_{n-1}), \ 1 \leq i \leq n, \ i \neq n-1 \\ (a_1, \cdots, a_{n-1}), \ i = 0 \end{cases}$$

where for $1 \leq i \leq n-2$, $(x_1, \cdots, x_{n-1})$ is the $(n-1)$-tuple obtained from $(a_1, \cdots, a_{n-1})$ by replacing a_i by a'_n, and for $i = n$, $(x_1, \cdots, x_{n-1}) = (a_1, \cdots, a_{n-2}, a'_n)$. Further, we have $\theta^{(n-1)}$ is the least (under $\geq$) in $\{(y_1^{(i)}, \cdots, y_{n-1}^{(i)}) \uparrow, 0 \leq i \leq n, \ i \neq n-1\}$.
(c). Given $\theta, w \in W$, say $\theta = (a_1 \cdots a_{2n})$, $w = (b_1 \cdots b_{2n})$, we have (with notations as in (b) above)

$$w^{(n-1)} \succeq \theta^{(n-1)} \Leftrightarrow (b_1, \cdots, b_{n-1}) \uparrow \succeq (y_1^{(i)}, \cdots, y_{n-1}^{(i)}) \uparrow \text{ for some } i, 0 \leq i \leq n, \ i \neq n-1.$$

6.2. **Chevalley Basis.** For $1 \leq k \leq 2n$, set $k' = 2n+1-k$. The involution $\sigma : SL(2n) \to SL(2n), A \mapsto E({}^tA)^{-1}E$, induces an involution $\sigma : sl(2n) \to sl(2n), A \mapsto -E({}^tA)E$ In particular, we have, for $1 \leq i, j \leq 2n$, $\sigma(E_{ij}) = -E_{j'i'}$, where E_{ij} is the elementary matrix with1 at the (i,j)th place and 0 elsewhere; and for $1 \leq k \leq 2n, k' = 2n+1-k$. Further

$$\text{Lie } G = \{A \in \mathrm{sl}(2n) \mid E({}^tA)E = -A\}.$$

The Chevalley basis $\{H_{\epsilon_i - \epsilon_{i+1}}, 1 \leq i < n, H_{\epsilon_{n-1}+\epsilon_n}, X_{\pm(\epsilon_j - \epsilon_k)}, X_{\pm(\epsilon_j + \epsilon_k)}, 1 \leq j < k \leq n\}$ for Lie G may be given as follows:

$$H_{\epsilon_i - \epsilon_{i+1}} = E_{ii} - E_{i+1,i+1} + E_{(i+1)',(i+1)'} - E_{i'i'},$$
$$H_{\epsilon_{n-1}+\epsilon_n} = E_{n-1,n-1} + E_{n,n} - E_{n',n'} - E_{(n-1)',(n-1)'},$$
$$X_{\epsilon_j - \epsilon_k} = E_{jk} - E_{k'j'}, \ X_{\epsilon_j + \epsilon_k} = E_{jk'} - E_{kj'},$$
$$X_{-(\epsilon_j - \epsilon_k)} = E_{kj} - E_{j'k'}, \ X_{-(\epsilon_j + \epsilon_k)} = E_{k'j} - E_{j'k}.$$

Definition 6.3. Let $\phi = (a_1 \cdots a_d) \in W^d,\ 1 \leq d \leq n-2$. Further let $n-1, n \in \{a_1, \cdots, a_d\}$. Let $\tau = (b_1 \cdots b_d)$ be the element of W^d obtained from $(a_1 \cdots a_d)$ by replacing $n-1$ by n', and n by $(n-1)'$. In this situation, we say that τ is obtained from ϕ by a *Type II* operation.

Remark 6.4. *Type I* operation is defined exactly as in Definition **??**.

Proposition 6.5. *(cf. [?]) Let $\tau, \phi \in W^d,\ 1 \leq d \leq n-2,\ \tau \geq \phi$. Then (τ, ϕ) is an admissible pair if and only if either $\tau = \phi$, or τ is obtained from ϕ by a sequence of operations of Type I or II.*

6.6. **The G-module $V(\omega_d)$.** For $d = n-1, n$, $V(\omega_d)$ is the spin representation, and the extremal weight vectors, $q_\tau, \tau \in W^d$ form a basis for $V(\omega_d)$. For $1 \leq d \leq n-2$, we have $V(\omega_d) = \wedge^d V$ (here, $V = K^{2n}$), and the extremal weight vectors $\{q_\tau . \tau \in W^d\}$, say $\tau = (a_1 \cdots a_d)$ are given by

$$q_\tau = e_{a_1} \wedge \cdots \wedge e_{a_d}$$

.

Proposition 6.7. *(cf. [?]) Let $\beta \in R^+$.*

(1) *Let $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$. Then*

$$X_{-\beta} q_{id} = \begin{cases} 0, & \text{if } d < j \text{ or } d \geq k \\ \pm q_{s_\beta^{(d)}}, & \text{if } j \leq d < k. \end{cases}$$

(2) *Let $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$.*

(a) *If $k = n-1, n$, then*

$$X_{-\beta} q_{id} = \begin{cases} 0, & \text{if } d < j \\ \pm q_{s_\beta^{(d)}}, & \text{if } j \leq d. \end{cases}$$

(b) *If $k < n-1$, then*

$$X_{-\beta} q_{id} = \begin{cases} 0, & \text{if } d < j \\ \pm q_{s_\beta^{(d)}}, & \text{if } j \leq d < k \text{ or } d = n-1, n \\ \pm(\sum_{i=0}^{n-d} c_i q_{\theta_i, \delta_i} + a q_{\tau, \phi}), & \text{if } k \leq d < n-1 \end{cases}$$

where $\tau, \phi, \theta_i, \delta_i, 0 \leq i \leq n-d$ are defined in the same way as in Proposition **??**, and $c_i = \pm 2$ or ± 1 according as $i <$ or $\geq n-d-1$, and $a = 1$.

Theorem 6.8. *Let $w \in W$, and $\beta \in R^+$.*

(1) *Let $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$, or $\epsilon_j + \epsilon_k$, $k = n-1, n$, $1 \leq j < k$. Then $\beta \in N(w, e)$ if and only if $w \succeq s_\beta$.*

(2) *Let $\beta = \epsilon_j + \epsilon_k,\ j < k < n-1$. Then $\beta \in N(w, e)$ if and only if $w \succeq$ either s_β or $s_{\epsilon_j - \epsilon_n} s_{\epsilon_j + \epsilon_n} s_{\epsilon_k + \epsilon_{n-1}}$.*

Proof. If $\beta = \epsilon_j - \epsilon_k$, $1 \le j < k \le n$, or $\epsilon_j + \epsilon_k$, $k = n-1, n$, $1 \le j < k$, then the result follows from Proposition **??**, 1 and 2(a).

Let $\beta = \epsilon_j + \epsilon_k$, $1 \le j < k \le n-2$. If $w \succeq$ either s_β or $s_{\epsilon_j - \epsilon_n} s_{\epsilon_j+\epsilon_n} s_{\epsilon_k+\epsilon_{n-1}}$, then clearly $X_{-\beta} q_{id} \in V_w(\omega_d)$, for all $1 \le d \le n$, and hence $\beta \in N(w, e)$ (cf. Theorem **??**). Let now $\beta \in N(w, e)$, and let $w \not\succeq s_\beta$. We shall now show that $w \succeq s_{\epsilon_j - \epsilon_n} s_{\epsilon_j+\epsilon_n} s_{\epsilon_k+\epsilon_{n-1}}$. We have, $w^{(d)} \succeq s_\beta^{(d)}, j \le d < k$. This implies in particular that $w^{(j)} \succeq s_\beta^{(j)}$. Hence there exists an entry p in $\{a_1 \cdots a_j\}$ such that $p \ge k'$. Also, the facts that $w \not\succeq s_\beta$, $w^{(j)} \succeq s_\beta^{(j)}$ imply that

$$w^{(k)} \not\succeq s_\beta^{(k)}. \qquad (\dagger)$$

On the other hand, since $\beta \in N(w, e)$, we have $X_{-\beta} q_{id} \in V_w(\omega_d)$, $1 \le d \le n$ (cf. Theorem **??**) and hence we obtain $w^{(k)} \succeq \tau, \theta$, and δ, where $\tau = (12 \cdots j-1\, j+1 \cdots k\, j')$, $\theta = (12 \cdots j-1\, j+1 \cdots k-1\, n\, k')$, $\delta = (12 \cdots j-1\, j+1 \cdots k-1\, n'\, (n-1)')$ (cf. Proposition **??**, 2(b)). In particular, we obtain (since $w^{(k)} \succeq \tau$) that there exists an entry q in $\{a_1 \cdots a_k\}$ such that $q \ge j'$. We have in fact $q \in \{a_1 \cdots a_j\}$, and $q = p$ (otherwise, we would obtain $w^{(k)} \succeq s_\beta^{(k)}$ contradicting ($\dagger$)). Thus

$$w^{(j)} \succeq (12 \cdots j-1\, j+1\, j'). \qquad (1)$$

Further we obtain (since $w^{(k)} \succeq \theta, \delta$)

$$w^{(k)} \succeq (12 \cdots j-1\, j+1 \cdots k-1\, (n-1)'\, j') \qquad (2)$$

(note that $(12 \cdots j-1\, j+1 \cdots k-1\, (n-1)'\, j')$ is the smallest (under $\succeq$) element ϕ in W^k for the property that $\phi \succeq \tau, \theta$, and δ; note that eventhough the k-tuple $(12 \cdots j-1\, j+1 \cdots k-1 n'\, j') \ge \tau, \theta$, and δ, it is $\not\succeq \theta$ (since they have non-compatible analugous parts (cf. (IV) above))). Also, we have

$$w^{(d)} \succeq s_\beta^{(d)}, \; d = n-1, n. \qquad (3)$$

Now it is easily seen that $\xi = (12 \cdots j-1\, j' j+1 \cdots k-1\, (n-1)'\, k+1 \cdots n-2\, k'\, n')$ is the smallest (under $\succeq$)) element in W for the properties given in (**??**), (**??**), and (**??**) (note that eventhough $\eta := (12 \cdots j-1\, j' j+1 \cdots k-1\, (n-1)'\, k+1 \cdots n-2\, n'\, k')$ is the smallest (under $\ge$) element in W such that $\eta^{(j)} \ge (12 \cdots j-1\, j+1\, j')$, $\eta^{(k)} \ge (12 \cdots j-1\, j+1 \cdots k-1\, (n-1)'\, j')$, $\eta^{(n-1)} \ge (12 \cdots j-1\, j+1 \cdots k-1\, k+1 \cdots n-1\, n'\, k')$ $(= (s_\beta u_j)^{(n-1)}$ (cf. Remark **??**)), $\eta^{(n)} \ge s_\beta^{(n)}$, we have, $\eta^{(n-1)} \not\succeq s_\beta^{(n-1)}$ (since $\eta^{(n-1)}$ and $(s_\beta u_j)^{(n-1)}$ have non-compatible analugous parts)). From this it follows that $w \succeq s_{\epsilon_j - \epsilon_n} s_{\epsilon_j+\epsilon_n} s_{\epsilon_k+\epsilon_{n-1}}$ (note that $s_{\epsilon_j - \epsilon_n} s_{\epsilon_j+\epsilon_n} s_{\epsilon_k+\epsilon_{n-1}} = \xi$).

□

Proposition 6.9. *Let $\beta \in R^+$. Let $\beta = \epsilon_j + \epsilon_k$, $1 \leq j < k \leq n$, and $k \leq d \leq n-2$. Then $m(\omega_d - \beta) = n - d + 2$. In all other cases, we have, $m(\omega_d - \beta) = 0$ or 1.*

Proof. Fix $d, 1 \leq d \leq n$.
(1) Let $\beta = \epsilon_j - \epsilon_k, 1 \leq j < k \leq n$. If $d < j$ or $d \geq k$, then clearly $m(\omega_d - \beta) = 0$. Let then $j \leq d < k$. Then $q_{s_\beta^{(d)}}$ is the only vector in $\mathcal{B}_d$ of weight $\omega_d - \beta$.
(2) Let $\beta = \epsilon_j + \epsilon_k, 1 \leq j < k \leq n$. If $d < j$, then clearly $m(\omega_d - \beta) = 0$. Let then $j \leq d \leq n$. If $d = n-1, n$, or $k = n-1, n$, or $j \leq d < k$, then $q_{s_\beta^{(d)}}$ is the only vector in $\mathcal{B}_d$ of weight $\omega_d - \beta$. Let then $k \leq d \leq n-2$. Then it is easily checked (by weight considerations) that q_{θ_i,δ_i}, $0 \leq i \leq n-d$, $q_{\tau,\phi}$ (as in Proposition **??**, 2(b)) are the only vectors in $\mathcal{B}_d$ of weight $\omega_d - \beta$.
The required result follows from (1) and (2). □

Theorem 6.10. *Let $w \in W$, and $\beta \in R^+$. Then $\beta \in N(w, e)$ if and only if $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$.*

Proof. The required result follows from Proposition 6.7, Theorem **??** and (1), (2) in the proof of Proposition **??**. □

Remark 6.11. Let $\beta \in R^+$, $w \in W$.
If $\beta = \epsilon_j - \epsilon_k$, $j < k \leq n$, or $\epsilon_j + \epsilon_k$, $k = n-1, n$, then the three conditions in §**??** are equivalent.
For all $\beta \in R^+$, $w \in W$, the condition that $\beta \in N(w, e)$ is equivalent to the condition $m_w(\omega_d - \beta) = m(\omega_d - \beta)$, for all $1 \leq d \leq n$ (cf. Theorem **??**).
The condition that $w \geq s_\beta$ need not be equivalent to the condition that $\beta \in N(w, e)$. For example, take $\beta = \epsilon_j + \epsilon_k$, $j < k < n-1$, and $w = s_{\epsilon_j - \epsilon_n} s_{\epsilon_j + \epsilon_n} s_{\epsilon_k + \epsilon_{n-1}}$; we have (cf. Theorem **??**, (2)), $\beta \in N(w, e)$, but $w \not\geq s_\beta$.

Remark 6.12. The statement (in Theorem 1 in [**?**]) that $\beta \in N(w, e)$ if and only if $m_w(\rho - \beta) = m(\rho - \beta)$ is incorrect. Although, there is a similarity in the statement of Theorem 1 in [**?**] and those in Theorems **??**, **??** and **??**, the property that $m_w(\rho - \beta) = m(\rho - \beta)$ seems to be stronger than the property that $\beta \in N(w, e)$. For example, consider $G = Sp(6)$, $w = s_{2\epsilon_1}$. Then for $\beta = \epsilon_1 + \epsilon_2$, we have, $X_{-\beta} q_{id} \in V_{w,\rho}$, but $m_w(\rho - \beta) \neq m(\rho - \beta)$ (we have, $m(\rho - \beta) = 6$, $m_w(\rho - \beta) = 5$.)

References

[1] N. Bourbaki, *Groupes et Algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968.

[2] V. Lakshmibai, *Geometry of G/P - VI*, J. Alg., Vol. **108** (1987), 355-402.

[3] V. Lakshmibai, *Bases for quantum Demazure modules*, CMS Conference Proceedings, vol **16**, 199-216, Proceedings of the conference on " Representations of Groups: Lie, Algebraic, Finite, and Quantum", Banff, Alberta, June 1994.

[4] V. Lakshmibai, *Tangent spaces to Schubert Varieties*, Math. Research Letters **2** (1995), 473-477.

[5] V. Lakshmibai and C.S. Seshadri, *Geometry of G/P - V*, J. Alg., Vol. **100** (1986), 462-557.

[6] V. Lakshmibai and C.S. Seshadri *Geometry of G/P - II*, Proc. Ind. Acad. Sci., 87A (1978), 1-54

[7] V. Lakshmibai and C.S. Seshadri, *Singular locus of a Schubert variety*, Bull. A.M.S., **11** (1984), 363-366.

[8] P. Polo, *On Zariski tangent spaces of Schubert varieties, and a proof of a conjecture of Deodhar*, Indag. Math., **5**, 1994, 483-493.

[9] R. Proctor, *Classical Bruhat orders are lexicographic shellable*, J. Alg **77**, 1982, 104-126.

Department of Mathematics, Northeastern University, Boston, MA 02115

E-mail address: `lakshmibai@neu.edu`