SINGULAR LOCI OF VARIETIES OF COMPLEXES

V. LAKSHMIBAI[†]

Introduction

Let $V_1, \dots, V_{h+1}$ be vector spaces over an arbitrary field K with respective dimensions $n_1, \dots, n_{h+1}$. Let Z be the affine space of all h-tuples of linear maps $(f_1, \dots, f_h) : V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{h-1}} V_h \xrightarrow{f_h} V_{h+1}$. If we endow each V_i with a basis, we get $V_i \cong K^{n_i}$ and $Z \cong M(n_2 \times n_1) \times \cdots \times M(n_{h+1} \times n_h)$, where $M(l \times m)$ denotes the affine space of matrices over K with l rows and m columns. Let V be the variety of complexes, namely, the subvariety of Z consisting of $\{(A_1, A_2, \cdots, A_h) \mid A_i A_{i-1} = 0,\ 2 \le i \le h\}$. It is shown in [?] (see also [?]) that each irreducible component of V is isomorphic to the *opposite cell* in a Schubert variety in G/Q, where $G = SL(n)$, $n = n_1 + \cdots + n_{h+1}$, and Q is a certain parabolic sub group. Let C be an irreducible component of V. In this paper, we determine the singular locus of C (cf. Theorem **??**), for the case $h = 2$. Let $X(w)\,(\subset G/B)$ be the Schubert variety associated to C. We further show (Theorem **??**) that the conjecture of [?] holds for $X(w)$. These w's turn out to be non-vexillary (cf. [?]), i.e., the corresponding permutations involve the pattern c, d, a, b, where $a < b < c < d$.

1. Quiver varieties.

Let $V_i,\ 1 \le i \le h+1$, Z be as in the Introduction. Let $\mathbf{n} = (\mathbf{n_1}, \dots, \mathbf{n_{h+1}})$. The group $G_{\mathbf{n}} = GL(n_1) \times \cdots \times GL(n_{h+1})$ acts on Z by

$$(g_1, g_2, \cdots, g_{h+1}) \cdot (f_1, f_2, \cdots, f_h) = (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \cdots, g_{h+1} f_h g_h^{-1}).$$

Now, let $\mathbf{r} = (\mathbf{r_{ij}})_{\mathbf{1 \le i \le j \le h+1}}$ be an array of non-negative integers with $r_{ii} = n_i$; and define $r_{ij} = 0$ for any indices other than $1 \le i \le j \le h+1$. Define the sets

$$Z^\circ(\mathbf{r}) = \{(\mathbf{f_1}, \cdots, \mathbf{f_h}) \in \mathbf{Z} \ \mid\ \forall\, \mathbf{i} < \mathbf{j},\ \ \text{rank } (\mathbf{f_{j-1}} \cdots \mathbf{f_i} : \mathbf{V_i} \to \mathbf{V_j}) = \mathbf{r_{ij}}\},$$

$$Z(\mathbf{r}) = \overline{\mathbf{Z}^\circ(\mathbf{r})}.$$

(These sets might be empty for a bad choice of $\mathbf{r}$.)

Proposition 1.1. *(see [?] for example)*

(1) *The $G_{\mathbf{n}}$-orbits of Z are exactly the sets $Z^\circ(\mathbf{r})$ for $\mathbf{r} = (\mathbf{r_{ij}})$ with $r_{ij} - r_{i,j+1} - r_{i-1,j} + r_{i-1,j+1} \ge 0, \quad \forall\ 1 \le i < j \le h+1$.*

(2) $Z(\mathbf{r}) = \{(\mathbf{f_1}, \cdots, \mathbf{f_h}) \in \mathbf{Z} \ \mid\ \forall\, \mathbf{i} < \mathbf{j},\ \ \textit{rank } (\mathbf{f_{j-1}} \cdots \mathbf{f_i} : \mathbf{V_i} \to \mathbf{V_j}) \le \mathbf{r_{ij}}\}$.

2. The Schubert variety $X_Q(w)$.

Given $\mathbf{n} = (\mathbf{n_1}, \cdots, \mathbf{n_h})$, for $1 \le i \le h+1$, let $a_i = n_1 + \cdots + n_i$, $a_0 = 0$, $n = n_1 + \cdots + n_{h+1}$. For positive integers $i \le j$, we shall frequently use the notations

$$[i, j] = \{i, i+1, \dots, j\},\ [i] = \{i\},\ [0] = \{\},\ \text{the empty set .}$$

Consider $GL(n)$, its subgroup B of upper-triangular matrices, and the parabolic subgroup $Q = \{(a_{ij}) \in GL(n) \mid a_{ij} = 0 \text{ whenever } j \le a_k < i \text{ for some } k\}$. We have, $Q =$

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$P_{a_1} \cap \cdots \cap P_{a_h}$, where for $1 \le i \le n-1$, P_i denotes the maximal parabolic sub group associated to the simple root α_i, the simple roots being indexed as in [?].

2.1. **The partially ordered set $I_{a_1,\dots,a_h}$.** Let

$$I_{a_1,\dots,a_h} = \{(\underline{i}_1,\dots,\underline{i}_h) \in I_{a_1,n} \times \cdots \times I_{a_h,n} \mid \underline{i}_t \subset \underline{i}_{t+1} \text{ for all } 1 \le t \le h-1\},$$

where for $d < n$, $I_{d,n} := \{\underline{i} = (i_1,\dots,i_d) \mid 1 \le i_1 < \cdots < i_d \le n\}$. Then it is easily seen that W^Q, the set of minimal representatives of W/W_Q (W being the symmetric group S_n) may be identified with $I_{a_1,\dots,a_k}$.

The partial order on the set of Schubert varieties in G/Q (given by inclusion) induces a partial order $\ge$ on $I_{a_1,\dots,a_h}$, namely, for $\mathbf{i} = (\underline{\mathbf{i}}_1,\dots,\underline{\mathbf{i}}_h)$, $\mathbf{j} = (\underline{\mathbf{j}}_1,\dots,\underline{\mathbf{j}}_h) \in \mathbf{I}_{\mathbf{a}_1,\dots,\mathbf{a}_h}$, $\mathbf{i} \ge \mathbf{j} \iff \underline{\mathbf{i}}_t \ge \underline{\mathbf{j}}_t$ for all $1 \le t \le h$.

2.2. **The opposite big cell in G/Q.** Let $Q = \cap_{t=1}^h P_{a_t}$ be as above. Let R (resp. R_Q) denote the root system of G (resp. Q). Denote by O^- the subgroup of G generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$ (here, for a root α, U_α denotes the 1-dimensional unipotent subgroup of G associated to α). Then O^- consists of the elements of G of the form

$$\begin{pmatrix} I_1 & 0 & 0 & \cdots & 0 & 0 \\ * & I_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \cdots & I_h & 0 \\ * & * & * & \cdots & * & I_a \end{pmatrix},$$

where I_t is the identity matrix of size $a_t - a_{t-1}$, $1 \le t \le h$, I_a is the identity matrix of size a, where $a = n - a_h$, and if $x_{ml} \ne 0$, with $m \ne l$, then $m > a_t$, $l \le a_t$ for some t, $1 \le t \le h$. Further, the restriction of the canonical morphism $f : G \to G/Q$ to O^- is an open immersion, and $f(O^-) \simeq B^- e_{\mathrm{id},Q}$. Thus $B^- e_{\mathrm{id},Q}$ (*the opposite big cell* in G/Q) gets identified with O^-.

We shall index the Schubert varieties in G/Q by W^Q. For $\tau \in W^Q$, we shall denote the associated Schubert variety by $X_Q(\tau)$; we set $Y_Q(\tau) = X_Q(\tau) \cap O^-$, *the opposite cell* in $X_Q(\tau)$. Note that $Y_Q(\tau)$ is a non-empty closed subvariety of O^-.

2.3. **The variety $X_Q(w)$.** Given $\mathbf{r} = (\mathbf{r_{ij}})_{\mathbf{1 \le i \le j \le h+1}}$, define $w \in W^Q$ by

$$w^{(a_i)} = \{\underbrace{1 \dots a_{i-1}}_{a_{i-1}} \; \underbrace{\dots\dots . a_i}_{r_{ii}-r_{i,i+1}} \; \underbrace{\dots\dots . a_{i+1}}_{r_{i,i+1}-r_{i,i+2}} \; \underbrace{\dots\dots . a_{i+2}}_{r_{i,i+2}-r_{i,i+3}} \; \cdots \; \underbrace{\dots\dots . n}_{r_{i,h+1}}\}$$

where we use the visual notation $\underbrace{\cdots\cdots a}_{b} = [a-b+1, a]$.

2.4. **The map f.** Let $z = (A_1, A_2, \cdots, A_h) \in Z$. Define $f : Z \to \mathcal{O}^-$ by

$$f(z) = \begin{pmatrix} I_1 & 0 & 0 & 0 & \cdots \\ A_1 & I_2 & 0 & 0 & \cdots \\ A_2A_1 & A_2 & I_3 & 0 & \cdots \\ A_3A_2A_1 & A_3A_2 & A_3 & I_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \pmod{Q}.$$

Denote a generic element of $Z = M(n_2 \times n_1) \times \cdots \times M(n_{h+1} \times n_h)$ by $(A_1,\dots,A_h)$, so that the coordinate ring of Z is the polynomial ring in the entries of all the matrices A_i. Let $\mathcal{J}(\mathbf{r}) \subset \mathbf{K}[\mathbf{Z}]$ be the ideal generated by the determinantal conditions implied by the definition of $Z(\mathbf{r})$:

$$\mathcal{J}(\mathbf{r}) = \left\langle \det(\mathbf{A_{j-1}A_{j-2}} \cdots \mathbf{A_i})_{\lambda\times\mu} \;\middle|\; \begin{array}{c} j > i,\ \lambda \subset [n_j],\ \mu \subset [n_i] \\ \#\lambda = \#\mu = r_{ij}+1 \end{array} \right\rangle.$$

Theorem 2.5. *(cf.[?])*

(1) *$\mathcal{J}(\mathbf{r})$ is a prime ideal and is the vanishing ideal of $Z(\mathbf{r}) \subset \mathbf{Z}$.*
(2) *The restriction $f\,|_{Z(\mathbf{r})}$ defines an isomorphism of $Z(\mathbf{r})$ onto $Y_Q(w)$.*

3. Variety of complexes.

Let V be the variety of complexes, namely, the subvariety of Z consisting of $\{(A_1, A_2, \cdots, A_h) \mid A_i A_{i-1} = 0,\ 2 \le i \le h\}$. We have $V = \cup_{\mathbf{r}}\ Z(\mathbf{r})$, where $r_{ij} = 0,\ j \ne i, i+1$. Let us denote $k_i = r_{ii+1},\ \mathbf{k} = (\mathbf{k_1}, \mathbf{k_2}, \cdots, \mathbf{k_h})$, and $V(\mathbf{k}) = \mathbf{Z}(\mathbf{r})$. Let

$$A_{\mathbf{n}} = \{\mathbf{k} = (\mathbf{k_1}, \mathbf{k_2}, \cdots, \mathbf{k_h}) \mid \mathbf{k_i} \le \ \min\{\mathbf{n_i}, \mathbf{n_{i+1}}\},\ \mathbf{1} \le \mathbf{i} \le \mathbf{h},\ \mathbf{k_{i-1}} + \mathbf{k_i} \le \mathbf{n_i},\ \mathbf{2} \le \mathbf{i} \le \mathbf{h}\}.$$

Then $V = \cup_{\mathbf{k} \in \mathbf{A_n}}\ V(\mathbf{k})$.

3.1. **A partial order on $\{(k_1, k_2, \cdots, k_h)\}$.** The partial order on the set $\{V(\mathbf{k}),\ \mathbf{k} \in \mathbf{A_n}\}$ of $G_{\mathbf{n}}$-orbit closures in V given by inclusion induces a partial order $\ge$ on $A_{\mathbf{n}}$, namely, for $\mathbf{k} = (\mathbf{k_1}, \mathbf{k_2}, \cdots, \mathbf{k_h}),\ \mathbf{k}' = (\mathbf{k'_1}, \mathbf{k'_2}, \cdots, \mathbf{k'_h})$ in $A_{\mathbf{n}}$, $\mathbf{k} \ge \mathbf{k}' \iff \mathbf{k_t} \ge \mathbf{k'_t},\ \mathbf{1} \le \mathbf{t} \le \mathbf{h}$.

Theorem 3.2. (1) $V(\mathbf{k}) \simeq \mathbf{Y_Q}(\mathbf{w})$
(2) *$\{V(\mathbf{k}) \mid \mathbf{k}$ is a maximal element of $\mathbf{A_n}\}$ gives the set of all irreducible components of V.*

Proof. Assertion (1) follows from Theorem ??. Assertion (2) follows from the fact that $Z(\mathbf{r}') \subset \mathbf{Z}(\mathbf{r}) \iff \mathbf{r'_{ij}} \le \mathbf{r_{ij}},\ \mathbf{1} \le \mathbf{i} < \mathbf{j} \le \mathbf{h} + \mathbf{1}$. □

Remark 3.3. The results in Theorem ?? are also proved in [?].

Let Y denote the affine space (considered as a subvariety of O^-) given by

$$Y = \left\{ \begin{pmatrix} I_1 & 0 & 0 & 0 & \cdots \\ A_1 & I_2 & 0 & 0 & \cdots \\ 0 & A_2 & I_3 & 0 & \cdots \\ 0 & 0 & A_3 & I_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \right\}.$$

For the rest of the paper we shall identify V with the subvariety of Y consisting of $\{(A_1, A_2, \cdots, A_h) \mid A_i A_{i-1} = 0,\ 2 \le i \le h\}$.

Lemma 3.4. *(cf. [?], [?]) Let $\mathbf{k} \in \mathbf{A_n}$.*
(1) *$\dim V(\mathbf{k}) = \sum_{\mathbf{1} \le \mathbf{i} \le \mathbf{h}+\mathbf{1}} (\mathbf{n_i} - \mathbf{k_i})\,(\mathbf{k_{i-1}} + \mathbf{k_i})$, where $k_0 = k_{h+1} = 0$.*
(2) *$\mathrm{codim}_Y V(\mathbf{k}) = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{h}} \mathbf{c_i} + \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{h}-\mathbf{1}} \mathbf{k_i}\mathbf{k_{i+1}}$, where $c_i = (n_{i+1} - k_i)\,(n_i - k_i)$.*

4. Singular locus of $V(k_1, k_2)$.

In this section, we take $h = 2$, and determine the singular locus of $V(\mathbf{k})$. Given $\mathbf{n} = (\mathbf{n_1}, \mathbf{n_2}, \mathbf{n_3})$, we fix (k_1, k_2) such that $k_i \le \ \min\{n_i, n_{i+1}\},\ i = 1, 2,\ k_1 + k_2 \le n_2$. For $\mathbf{r} = (\mathbf{k_1}, \mathbf{k_2})$, we shall denote $O(k_1, k_2) = Z^0(\mathbf{r}),\ \mathbf{V}(\mathbf{k_1}, \mathbf{k_2}) = \mathbf{Z}(\mathbf{r})(= \mathbf{V}(\mathbf{k}))$.

Lemma 4.1. *Let $c_i = (n_{i+1} - k_i)\,(n_i - k_i),\ i = 1, 2$. We have*
(1) $c_1 + k_1 k_2 - n_1 k_2 = (n_1 - k_1)(n_2 - k_1 - k_2)$.
(2) $c_2 + k_1 k_2 - n_3 k_1 = (n_3 - k_2)(n_2 - k_1 - k_2)$.

Proof. The assertions follow from the definition of c_1, c_2. □

4.2. **The Jacobian matrix $\mathcal{J}$.** Let $X_i = (x^{(i)}_{kl}),\ i = 1,2$ denote the matrix of variables of size $n_2 \times n_1$ (resp. $n_3 \times n_2$). Let $\mathcal{M}_\rangle = \{$ all $\|_\rangle + \infty$ minors of $\mathcal{X}_\rangle\},\ \rangle = \infty, \in$, $f_{rs} = (r,s)$ -th entry in $X_2X_1,\ 1 \le r \le n_3,\ 1 \le s \le n_1$. Let $\mathcal{J}$ be the Jacobian matrix of V (V considered as a subvariety of Y, Y being as in §3). We shall index the rows of $\mathcal{J}$ by $\mathcal{M}_\rangle,\ \rangle = \infty, \in,\ \{\nabla_\int,\ \infty \le \nabla \le \backslash_\ni,\ \infty \le \int \le \backslash_\infty$, and the columns by $\{x^{(i)}_{kl},\ 1 \le k \le n_{i+1},\ 1 \le l \le n_i,\ i = 1,2\}$. Let $M \in \mathcal{M}_\rangle,\ \rangle = \infty, \in$. The entry in the $(M, x^{(j)}_{kl})$ -th position in $\mathcal{J}$ is non-zero if and only if $j = i$ and $x^{(i)}_{kl}$ is an entry in M, in which case it is equal to $\pm$ the k_i-minor of M obtained by deleting the row and column through $x^{(i)}_{kl}$. Also, for a variable $x^{(1)}_{ij}$, the only relevant functions are $f_{mj},\ 1 \le m \le n_3$. The $(f_{mj}, x^{(1)}_{ij})$-th entry in $\mathcal{J}$ is $x^{(2)}_{mi}$. Similarly, for a variable $x^{(2)}_{ij}$, the only relevant functions are $f_{im},\ 1 \le m \le n_1$. The $(f_{im}, x^{(2)}_{ij})$-th entry in $\mathcal{J}$ is $x^{(1)}_{jm}$. From this it follows that given $j,\ 1 \le j \le n_1$, the block in $\mathcal{J}$ with row indices given by $f_{rs},\ 1 \le r \le n_3$ (for a fixed s) and column indices given by the variables in the j-th column of X_1 is the zero block if $s \ne j$, and for $s = j$, it is simply X_2. Similarly, given $j,\ 1 \le j \le n_3$, the block in $\mathcal{J}$ with row indices given by $f_{sr},\ 1 \le r \le n_1$ (for a fixed s) and column indices given by the variables in the j-th row of X_2 is the zero block if $s \ne j$, and for $s = j$, it is simply ${}^tX_1$. Given $x \in V$, let $\mathcal{J}_\S$ denote $\mathcal{J}$ evaluated at x.

Lemma 4.3. *Let $x \in O(t_1,t_2)$. Let $\mathcal{J}_\prime$ be the submatrix of $\mathcal{J}_\S$ consisting of the rows indexed by $\{f_{rs},\ 1 \le r \le n_3,\ 1 \le s \le n_1\}$. Then rank$J_0 = n_1t_2 + n_3t_1 - t_1t_2$.*

Proof. Let $x = (R_1,R_2)$. We may take x to be the point of $O(t_1,t_2)$ given as follows: the right hand bottom block of size t_1 in R_1 is Id_{t_1} and the rest of the entries in R_1 are zero while the left hand top block of size t_2 in R_2 is Id_{t_2} and the rest of the entries in R_2 are zero. Note that rank$R_1 = t_1$, rank$R_2 = t_2$, $R_2R_1 = 0$ (since $n_2 \ge t_1 + t_2$). We shall denote by M_1 (resp. M_2) right hand bottom block of size t_1 in R_1 (resp. the left hand top block of size t_2 in R_2).

Let $\mathcal{B}_\rangle = \{|\rangle,\ \infty \le | \le \sqcup_\in\},\ \infty \le \rangle \le \backslash_\infty$, and $\mathcal{C}_\rangle = \{\rangle|,\ \infty \le | \le \backslash_\infty\},\ \sqcup_\in + \infty \le \rangle \le \backslash_\ni$. We shall index the rows of $\mathcal{J}_\prime$ as $\mathcal{B}_\infty, \mathcal{B}_\in, \cdots, \mathcal{B}_{\backslash_\infty},\ \mathcal{C}_{\sqcup_\in+\infty}, \mathcal{C}_{\sqcup_\in+\in}, \cdots, \mathcal{C}_{\backslash_\ni}$. Let J_1 (resp. J_2) be the submatrix of $\mathcal{J}_\prime$ consisting of the rows of $\mathcal{J}_\prime$ indexed by $\mathcal{B}_\infty, \mathcal{B}_\in, \cdots, \mathcal{B}_{\backslash_\infty}$ (resp. $\mathcal{C}_{\sqcup_\in+\infty}, \mathcal{C}_{\sqcup_\in+\in}, \cdots, \mathcal{C}_{\backslash_\ni}$).

We have (cf. §**??**), given $j,\ 1 \le j \le n_1$, the block in J_1 with row indices given by $\mathcal{B}_\|$ and column indices given by the variables in the j-th column of R_1 is the zero block if $k \ne j$, and for $k = j$, it is simply the submatrix of R_2 with row indices given by $1, 2, \cdots, t_2$, and column indices given by $1, 2, \cdots, n_2$ (and this block has rank t_2); similarly the block in J_2 with row indices $\{f_{rk},\ t_2 + 1 \le r \le n_3\}$ (for a fixed k) and column indices given by the variables in the j-th column of $R_1,\ 1 \le i \le n_1$ is the zero block if $k \ne j$, and for $k = j$, it is simply the submatrix of R_2 consisting of the last $(n_3 - t_2)$ rows of R_2 and hence is the zero block by our choice of R_2. Hence we obtain

$$\mathrm{rank}\mathcal{J}_\prime = \mathrm{rank}\mathcal{J}_\infty + \mathrm{rank}\mathcal{J}_\in.$$

From the discussion above, it follows easily that rank$J_1 = n_1t_2$. and rankJ_2 equals the rank of the submatrix of J_2 consisting of the columns indexed by $x^{(2)}_{ij},\ 1 \le i \le n_3,\ 1 \le j \le n_2$, and rank$J_2 = (n_3 - t_2)t_1$. Hence rank$J_0 = n_1t_2 + (n_3 - t_2)t_1 = n_1t_2 + n_3t_1 - t_1t_2$. □

Proposition 4.4. *Let $x = (A_1,A_2) \in V(k_1,k_2)$.*

(1) *Let rank $A_1 = k_1$, rank $A_2 < k_2$. Then x is a smooth point of $V(k_1,k_2)$ $\Longleftrightarrow$ either $n_3 = k_2$ or $n_2 = k_1 + k_2$.*
(2) *Let rank $A_2 = k_2$, rank $A_1 < k_1$. Then x is a smooth point of $V(k_1,k_2)$ $\Longleftrightarrow$ either $n_1 = k_1$ or $n_2 = k_1 + k_2$.*
(3) *Let rank $A_1 = k_1 - 1$, rank $A_2 = k_2 - 1$. Then x is a singular point of $V(k_1,k_2)$.*

Proof. Let $x = (A_1,A_2)$, and as above, let $\mathcal{J}_\S$ denote $\mathcal{J}$ evaluated at x.

(1) Let rank$A_2 = t_2$. Then $x \in O(k_1, t_2)$. We may take x to be the point of $O(k_1, t_2)$ given as follows: the right hand bottom block of size k_1 in A_1 is Id_{k_1} and the rest of the entries in A_1 are zero, while the left hand top block of size t_2 in A_2 is Id_{t_2} and the rest of the entries in A_2 are zero. Note that rank$A_1 = k_1$, rank$A_2 = t_2$, $A_2A_1 = 0$ (since $n_2 - k_1 \geq k_2 > t_2$). We shall denote by M_1 (resp. M_2) right hand bottom block of size k_1 in A_1 (resp. the left hand top block of size t_2 in A_2). Let $\mathcal{B}_\infty$ denote the set of all variables in A_1 not appearing in any of the last k_1 rows and last k_1 columns of A_1. Let $\tau \in \mathcal{B}_\infty$, and let M_τ be the (k_1+1)-minor of A_1 obtained from M_1 by adding the row and column of A_1 through τ. Then the (M_τ, τ)-th entry in $\mathcal{J}_\S$ is equal to 1 ; also for $\sigma \in \mathcal{B}_\infty$, $\sigma \neq \tau$, the (M_τ, σ)-th entry in $\mathcal{J}_\S$ is 0. Further, for $\tau \notin \mathcal{B}_\infty$, the (M, τ)-th entry in $\mathcal{J}_\S$ is zero, for all $M \in \mathcal{M}_\infty$ (by our choice of A_1). Also, for any variable ϕ in X_2, the (M, ϕ)-th entry in $\mathcal{J}_\S$ is zero, for all $M \in \mathcal{M}_\in$ (since rank$A_2 < k_2$). Let J_1 be the submatrix of $\mathcal{J}_\S$ consisting of the columns of $\mathcal{J}_\S$ indexed by the variables which are the entries of the first $(n_1 - k_1)$ columns of X_1 and J_2 the submatrix of $\mathcal{J}_\S$ consisting of the remaining columns of $\mathcal{J}_\S$. Then clearly rank$\mathcal{J}_\S =$ rank$J_1 +$ rankJ_2 (by our choice of A_1).

Computation of rankJ_1. We shall index the columns of J_1 by starting with the variables in the first column of X_1, followed by the variables in the second column, ..., followed by the variables in the $(n_1 - k_1)$-th column of X_1. Let J_{1i} be the submatrix of J_1 consisting the columns of J_1 indexed by the variables in the i-th column of X_1, $1 \leq i \leq n_1 - k_1$. Then by our choice of A_1, A_2, the non-zero entries in J_{1i} occur at the (M_τ, τ)-th place, $\tau \in \{x^{(1)}_{1i}, x^{(1)}_{2i}, \cdots, x^{(1)}_{n_2-k_1\, i}\}$, $(f_{ji}, x^{(1)}_{ji})$-th place, $1 \leq j \leq t_2$ (and these entries are equal to 1; note that the $(f_{ji}, x^{(1)}_{ji})$-th is $x^{(2)}_{jj}$). But now the fact that $t_2 < n_2 - k_1$ implies that the first $(n_2 - k_1)$ columns of J_{1i} are the only non-zero columns of J_{1i} and these are clearly linearly independant. Hence we obtain rank$J_{1i} = n_2 - k_1$, $1 \leq i \leq n_1 - k_1$, and hence rank$J_1 = \sum_{i=1}^{n_1-k_1}$ rank$J_{1i} = (n_1 - k_1)(n_2 - k_1) = c_1$.

Computation of rankJ_2. By our choice of A_1, we have, rankJ_2 equals the rank of the submatrix J_3 of J_2 consisting of the rows indexed by $\{f_{rs},\ 1 \leq r \leq n_3,\ n_1 - k_1 + 1 \leq s \leq n_1\}$. Taking R_1 of Lemma **??** as the submatrix of A_1 consisting of the last k_1 columns of A_1 and R_2 as A_2, we have, rank$J_2 = k_1t_2 + n_3k_1 - k_1t_2 = n_3k_1$ (note that in Lemma **??**, $n_1 = k_1$ since R_1 has size $n_2 \times k_1$, and $t_1 = k_1$, since rank$R_1 = k_1$).

Hence we obtain

$$\text{rank } \mathcal{J}_\S = \rfloor_\infty + \backslash_\ni \|_\infty.$$

If $n_3 = k_2$ or $n_2 = k_1 + k_2$, then $n_3k_1 = c_2 + k_1k_2$ (cf. Lemma **??**, (2)). Hence rank $\mathcal{J}_\S = \rfloor_\infty + \rfloor_\in + \|_\infty\|_\in =$ codim$_\mathcal{Y}\mathcal{V}(\|_\infty, \|_\in)$, and hence x is a smooth point of $V(k_1, k_2)$.

Let $n_3 > k_2$ and $n_2 > k_1 + k_2$. We have, rank $\mathcal{J}_\S = \rfloor_\infty + \backslash_\ni\|_\infty = \rfloor_\infty + \rfloor_\in + \|_\infty\|_\in - (\backslash_\ni - \|_\in)(\backslash_\in - \|_\infty - \|_\in)$ (cf. Lemma **??**, (2)). This together with the hypothesis that $n_3 > k_2$, $n_2 > k_1 + k_2$ implies that rank $\mathcal{J}_\S < \rfloor_\infty + \rfloor_\in + \|_\infty\|_\in (=$ codim$_\mathcal{Y}\mathcal{V}(\|_\infty, \|_\in))$. Hence x is a singular point of $V(k_1, k_2)$.

(2) The proof of (2) is similar. Let rank$A_1 = t_1$. Then $x \in O(t_1, k_2)$. We may take x to be the point of $O(t_1, k_2)$ given as follows: the right hand bottom block of size k_2 in A_2 is Id_{k_2} and the rest of the entries in A_2 are zero while the left hand top block of size t_1 in A_1 is Id_{t_1} and the rest of the entries in A_1 are zero. Note that rank$A_1 = t_1$, rank$A_2 = k_2$, $A_2A_1 = 0$ (since $n_2 - k_2 \geq k_1 > t_1$). We shall denote by M_1 (resp. M_2) left hand top block of size t_1 in A_1 (resp. the right hand bottom block of size k_2 in A_2). Let $\mathcal{B}_\in$ denote the set of all variables in A_2 not appearing in any of the last k_2 rows and last k_2 columns of A_2. Let $\tau \in \mathcal{B}_\in$, and let M_τ be the (k_2+1)-minor of A_2 obtained from M_2 by adding the row and column of A_2 through τ. Then the (M_τ, τ)-th entry in $\mathcal{J}_\S$ is equal to 1 ; also for $\sigma \in \mathcal{B}_\in$, $\sigma \neq \tau$, the (M_τ, σ)-th entry in $\mathcal{J}_\S$ is 0. Let J_1 be the submatrix of $\mathcal{J}_\S$ consisting of the columns of $\mathcal{J}_\S$ indexed by the variables which are the entries of the first $(n_3 - k_2)$ rows of X_2 and J_2 the submatrix of $\mathcal{J}_\S$ consisting of the remaining columns of $\mathcal{J}_\S$. Then clearly rank$\mathcal{J}_\S =$ rank$J_1 +$ rankJ_2 (by our choice of A_2).

Computation of rankJ_1. We shall index the columns of J_1 by starting with the variables in the first row of X_2, followed by the variables in the second row, ..., followed by the variables in the (n_3-k_2)-th row of X_2. Let J_{1i} be the submatrix of J_1 consisting the columns of J_1 indexed by the variables in the i-th row of X_2, $1 \le i \le n_3-k_2$. Then by our choice of A_1, A_2, the non-zero entries in J_{1i} occur at the (M_τ, τ)-th place, $\tau \in \{x_{i1}^{(2)}, x_{i2}^{(2)}, \cdots, x_{i\,n_3-k_2}^{(2)}\}$, $(f_{ij}, x_{ij}^{(2)})$-th place, $1 \le j \le t_1$ (and these entries are equal to 1; note that the $(f_{ij}, x_{ij}^{(2)})$-th entry is $x_{jj}^{(1)}$). But now the fact that $t_1 < n_2 - k_2$ implies that the first (n_2-k_2) columns of J_{1i} are the only non-zero columns of J_{1i} and these are clearly linearly independant. Hence we obtain rank$J_{1i} = n_2 - k_2$, $1 \le i \le n_3 - k_2$, and hence rank$J_1 = \sum_{i=1}^{n_3-k_2}$ rank$J_{1i} = (n_2-k_2)(n_3-k_2) = c_2$.

Computation of rankJ_2. By our choice of A_2, we have, rankJ_2 equals the rank of the submatrix J_3 of J_2 consisting of the rows indexed by $\{f_{rs},\ n_3-k_2+1 \le r \le n_3,\ 1 \le s \le n_1\}$. Taking R_2 of Lemma **??** as the submatrix of A_2 consisting of the last k_2 rows of A_2 and R_1 as A_1, we have, rank$J_2 = n_1k_2 + k_2t_1 - t_1k_2 = n_1k_2$ (note that in Lemma **??**, $n_3 = k_2$ since R_2 has size $k_2 \times n_2$, and $t_2 = k_2$, since rank$R_2 = k_2$).

Hence we obtain

$$\text{rank } \mathcal{J}_\S = \rfloor_\in + \backslash_\infty \|_\in .$$

If $n_1 = k_1$ or $n_2 = k_1 + k_2$, then $n_1k_2 = c_1 + k_1k_2$ (cf. Lemma **??**, (1)). Hence rank $\mathcal{J}_\S = \rfloor_\infty + \rfloor_\in + \|_\infty\|_\in =$ codim$_\mathcal{Y}\mathcal{V}(\|_\infty, \|_\in)$, and x is a smooth point of $V(k_1,k_2)$.

If $n_1 > k_1$ and $n_2 > k_1 + k_2$, then rank $\mathcal{J}_\S <$ codim$_\mathcal{Y}\mathcal{V}(\|_\infty, \|_\in)$ (in view of Lemma **??**,(1)). Hence x is a singular point of $V(k_1,k_2)$.

(3) Let rank $A_1 = k_1 - 1$, rank $A_2 = k_2 - 1$. Let us denote $t_1 = k_1 - 1$, $t_2 = k_2 - 1$. Then $x \in O(t_1,t_2)$. We have, the block in $\mathcal{J}_\S$ corresponding to the rows indexed by $\mathcal{M}_\infty$ and columns indexed by the set of all variables of A_1 is the zero block (since rank $A_1 < k_1$), and the block in $\mathcal{J}_\S$ corresponding to the rows indexed by $\mathcal{M}_\in$ and columns indexed by the set of all variables of A_2 is the zero block (since rank$A_2 < k_2$). Hence we obtain that rank$\mathcal{J}_\S$ equals the rank of the submatrix $\mathcal{J}_\prime$ in $\mathcal{J}_\S$ (consisting of the rows indexed by $\{f_{rs},\ 1 \le r \le n_3,\ 1 \le s \le n_1\}$). Now Lemma **??** (with $R_1 = A_1$, $R_2 = A_2$) implies that rank $\mathcal{J}_\S = \backslash_\infty\sqcup_\in + \backslash_\ni\sqcup_\infty - \sqcup_\infty\sqcup_\in$. This implies codim$_Y V(k_1,k_2) -$ rk$\mathcal{J}_\S =$ $\rfloor_\infty + \rfloor_\in + \|_\infty\|_\in - (\backslash_\infty\sqcup_\in + \backslash_\ni\sqcup_\infty - \sqcup_\infty\sqcup_\in)$
$= c_1 + c_2 + k_1k_2 - [c_1 + k_1k_2 - (n_1-k_1)(n_2-k_1-k_2) + c_2 + k_1k_2 - (n_3-k_2)(n_2-k_1-k_2)] + n_1 - n_3 + (k_1-1)(k_2-1)$ (cf. Lemma **??**) $= (n_1+n_3-k_1-k_2)(n_2-k_1-k_2+1) + 1 > 0$ (note that $n_1 + n_3 - k_1 - k_2 = (n_1-k_1) + (n_3-k_2) \ge 0$. From this it follows that x is a singular point of $V(k_1,k_2)$. □

Theorem 4.5. *Let SingV(k_1,k_2) denote the singular locus of $V(k_1,k_2)$. We have*

$$\mathit{Sing}V(k_1,k_2) = \begin{cases} V(k_1-1,k_2-1), \text{ if either} & k_1+k_2 = n_2 \text{ or } k_1 = n_1,\ k_2 = n_3 \\ V(k_1-1,k_2), & k_1+k_2 < n_2,\ k_1 < n_1,\ k_2 = n_3 \\ V(k_1,k_2-1), & k_1+k_2 < n_2,\ k_2 < n_3,\ k_1 = n_1 \\ V(k_1-1,k_2) \cup V(k_1,k_2-1), & k_1+k_2 < n_2,\ k_1 < n_1,\ k_2 < n_3. \end{cases}$$

Proof. Let $x = (A_1,A_2) \in V(k_1,k_2)$. If rk $A_i = k_i$, $i = 1,2$, then $x \in Z^\circ(k_1,k_2)$ (= the $G_{\mathbf{n}}$-orbit through x), and hence x is a smooth point of $V(k_1,k_2)$ (note that $V(k_1,k_2) = \overline{Z^0(k_1,k_2)}$). Thus we obtain (using Proposition **??**, (3))

$$V(k_1-1,k_2-1) \subseteq \text{ Sing } V(k_1,k_2) \subseteq V(k_1-1,k_2) \cup V(k_1,k_2-1).$$

The result follows from this (in view of Proposition **??**). □

Lemma 4.6. *$V(k_1,k_2)$ being an irreducible component of V, we have, either $k_1+k_2 = n_2$ or $k_1 = n_1$, $k_2 = n_3$.*

Proof. Note that (k_1,k_2) is a maximal element of $A_{\mathbf{n}}$ (for the partial order on $A_{\mathbf{n}}$ defined in §**??**). We divide the proof into the following two cases.

Case 1. $n_2 \geq n_1 + n_3$.
In this case, there is a unique maximal choice for k_1, k_2, namely $k_1 = n_1,\ k_2 = n_3$.
Case 2. $n_2 < n_1 + n_3$.
If $n_2 \geq n_1$, then $(k_1, k_2) = (n_1 - i, n_2 - n_1 + i),\ i = 0, 1, \cdots, n_1$ are all the possible maximal choices for (k_1, k_2), and for all such choices, we have, $k_1 + k_2 = n_2$.
Similarly, if $n_2 \geq n_3$, then $(k_1, k_2) = (n_2 - n_3 + i, n_3 - i),\ i = 0, 1, \cdots, n_3$ are all the possible maximal choices for (k_1, k_2), and for all such choices, we have, $k_1 + k_2 = n_2$.
If $n_2 < n_1,\ n_2 < n_3$, then $(k_1, k_2) = (i, n_2 - i),\ i = 0, 1, \cdots, n_2$ are all the possible maximal choices for (k_1, k_2), and for all such choices, we have, $k_1 + k_2 = n_2$. □

Remark 4.7. From the proof of Lemma **??**, we have, if $n_2 \geq n_1 + n_3$, then $k_1 = n_1,\ k_2 = n_3$; if $n_2 < n_1 + n_3$, then $k_1 + k_2 = n_2$.

Theorem 4.8. *$V(k_1, k_2)$ being an irreducible component of V, we have, Sing$V(k_1, k_2) = V(k_1 - 1, k_2 - 1)$.*

Proof. We have (cf. Lemma **??**), either $k_1 + k_2 = n_2$ or $k_1 = n_1,\ k_2 = n_3$ from which the result follows (in view of Theorem **??**). □

5. One Application.

We preserve the notations of sections §3,4. Let $V(k_1, k_2)$ be an irreducible component of V, and let $w \in W^Q$ be the element such that $V(k_1, k_2) \cong Y_Q(w)$. In this section, using Theorem **??** we deduce results on Sing$X(w^{\max})$, $w^{\max}$ being the maximal representative in W of the coset wW_Q (note that $X(w^{\max})$ is the pull back of $X(w)$ under the canonical projection $G/B \to G/Q$).

5.1. **A conjecture on the irreducible components of a Schubert variety in $SL(n)/B$.** Let $G = SL(n)$. We recall (cf. [?]) a conjecture on the irreducible components of the singular locus of a Schubert variety.

For $\tau \in W (= S_n)$, let P_τ (resp. Q_τ) be the maximal element of the set of parabolic subgroups which leave $\overline{B\tau B}$ $(\subset G)$ stable under multiplication on the left (resp. right).

Definition 5.2. Given parabolic subgroups P, Q, we say that $\overline{B\tau B}$ is P-Q stable if $P \subset P_\tau$ and $Q \subset Q_\tau$.

The set F_η.

Let $\eta = (a_1 \ldots a_n) \in \mathcal{S}_\backslash$. Let E_η be the set of all $\tau_1 \leq \eta$ such that either 1) or 2) below holds.
1) There exist $i, j, k, m,\ 1 \leq i < j < k < m \leq n$, such that
 (*a*) $a_k < a_m < a_i < a_j$
 (*b*) if $\tau_1 = (b_1 \ldots b_n)$, then there exist $i', j', k', m',\ 1 \leq i' < j' < k' < m' \leq n$ such that $b_{i'} = a_k,\ b_{j'} = a_i,\ b_{k'} = a_m,\ b_{m'} = a_j$
 (*c*) if τ (resp. θ_1) is the element obtained from η (resp. τ_1) by replacing a_i, a_j, a_k, a_m respectively by a_k, a_i, a_m, a_j (resp. $b_{i'}, b_{j'}, b_{k'}, b_{m'}$ respectively by $b_{j'}, b_{m'}, b_{i'}, b_{k'}$), then $\tau_1 \geq \tau$ and $\theta_1 \leq \eta$.
2) There exist $i, j, k, m,\ 1 \leq i < j < k < m \leq n$, such that
 (*a*) $a_m < a_j < a_k < a_i$
 (*b*) if $\tau_1 = (b_1 \ldots b_n)$, then there exist $i', j', k', m',\ 1 \leq i' < j' < k' < m' \leq n$ such that $b_{i'} = a_j,\ b_{j'} = a_m,\ b_{k'} = a_i,\ b_{m'} = a_k$
 (*c*) if τ (resp. θ_1) is the element obtained from η (resp. τ_1) by replacing a_i, a_j, a_k, a_m respectively by a_j, a_m, a_i, a_k (resp. $b_{i'}, b_{j'}, b_{k'}, b_{m'}$ respectively by $b_{k'}, b_{i'}, b_{m'}, b_{j'}$), then $\tau_1 \geq \tau$ and $\theta_1 \leq \eta$.

Let $F_\eta = \{\tau \in E_\eta \mid \overline{B\tau B} \text{ is } P_\eta\text{-}Q_\eta \text{ stable}\}$.

Conjecture . *Let $X(\eta) = \overline{B\eta B}$ (mod B) $(\subset G/B)$. The singular locus of $X(\eta)$ is equal to $\cup_\lambda X(\lambda)$, where λ runs over the maximal (under the Bruhat order) elements of F_η.*

5.3. Let $\eta = (a_1 \dots a_n) \in \mathcal{S}_\setminus$. Let $\operatorname{Sing} X(\eta) \neq \emptyset$. Let (a, b, c, d) be four distinct entries in $\{1, \dots, n\}$ such that $a < b < c < d$. An occurence in η of the form d, b, c, a, where $d = a_i$, $b = a_j$, $c = a_k$, $a = a_m$, $i < j < k < m$, will be referred to as a *Type I bad pattern in* η. An occurance in η of the form (c, d, a, b), where $c = a_i$, $d = a_j$, $a = a_k$, $b = a_m$, $i < j < k < m$, will be referred to as a *Type II bad pattern in* η. Recall (cf. [?]) that $\operatorname{Sing} X(\eta) \neq \emptyset$ if and only if there is a bad pattern of Type I or II in η.

5.4. **Application to Schubert varieties.** Let $V(k_1, k_2)$ be an irreducible component of V. Let $a_1 = n_1$, $a_2 = n_1 + n_2$, $a_3 = n_1 + n_2 + n_3 (= n)$, $G = SL(n)$, and $Q = P_{a_1} \cap P_{a_2}$. We have (cf. Theorem **??**) $V(k_1, k_2) \simeq Y_Q(w)$, $V(k_1 - 1, k_2 - 1) \simeq Y_Q(\theta)$, where $w, \theta \in W^Q$ are given by

$$w^{(a_1)} = (\underbrace{\dots a_1}_{a_1 - k_1} \ \underbrace{\dots . a_2}_{k_1}),\ w^{(a_2)} = (\underbrace{1 \dots a_1}_{a_1} \ \underbrace{\dots . a_2}_{n_2 - k_2} \ \underbrace{\dots . a_3}_{k_2}),$$

$$\theta^{(a_1)} = (\underbrace{\dots a_1}_{a_1 - k_1 + 1} \ \underbrace{\dots . a_2}_{k_1 - 1}),\ \theta^{(a_2)} = (\underbrace{1 \dots a_1}_{a_1} \ \underbrace{\dots . a_2}_{n_2 - k_2 + 1} \ \underbrace{\dots . a_3}_{k_2 - 1}).$$

5.5. We have the following two cases (cf. proof of Lemma **??**):
Case1. $n_2 \geq n_1 + n_3$.
In this case, we have $k_1 = n_1$, $k_2 = n_3$ (cf. Remark **??**). Hence

$$w^{(a_1)} = ([x, a_2]),\ w^{(a_2)} = ([1, a_1], [x', a_2], [y, a_3]),$$

$$\theta^{(a_1)} = ([n_1], [x + 1, a_2]),\ \theta^{(a_2)} = ([1, a_1], [x' - 1, a_2], [y + 1, a_3]),$$

where $x = a_2 + 1 - k_1 (= n_2 + 1)$, $x' = a_2 + 1 - (n_2 - k_2) (= n_1 + n_3 + 1)$, $y = a_3 + 1 - k_2 (= a_2 + 1)$.
Case 2. $n_2 < n_1 + n_3$.
In this case, we have, $k_1 + k_2 = n_2$ (cf. Remark **??**). Hence

$$w^{(a_1)} = ([p, a_1], [x, a_2]),\ w^{(a_2)} = ([1, a_1], [x, a_2], [y, a_3]),$$

$$\theta^{(a_1)} = ([p - 1, a_1], [x + 1, a_2]),\ \theta^{(a_2)} = ([1, a_1], [x - 1, a_2], [y + 1, a_3]),$$

where $p = a_1 + 1 - (n_1 - k_1)$, $x = a_2 + 1 - k_1 (= n_1 + k_2 + 1)$, $y = a_3 + 1 - k_2$.

Remark 5.6. Note that θ is the unique maximal element τ in W^Q such that $\tau \leq w$ and $\tau^{(a_i)}(a_i - k_i + 1) \leq a_i$, $i = 1, 2$ (here, $\tau^{(a_i)}(t)$ denotes the entry at the t-th place in $\tau^{(a_i)}$).

5.7. Let Q be as above. As above, we shall denote an element τ in W^Q as $\tau = (\tau^{(a_1)}, \tau^{(a_2)})$. Let $\tau^{\max}$ denote the maximal representative in W of the coset τW_Q. Note that $X(\tau^{\max}))$ is the inverse image of $X_Q(\theta))$ under the canonical projection $G/B \to G/Q$; also, $\tau^{\max}$ as a permutation is given by writing the entries in $\tau^{(a_1)}$ in descending order, followed by the entries in $\tau^{(a_2)} \setminus \tau^{(a_1)}$ in descending order, followed by the entries in $\{1, \cdots, n\} \setminus \tau^{(a_2)}$ in descending order. In the sequel, we shall refer to these as the I,II,III blocks respectively of $\tau^{\max}$. Also, in the sequel, for an interval $[a, b]$, $[a, b]_-$ shall denete the entries of $[a, b]$ written in descending order. Thus, in Case 1 of **??**, we have (with notations as in **??**),
I block in $w^{\max} = [n_2 + 1, a_2]_-$, I block in $\theta^{\max} = ([n_2 + 2, a_2]_-, [n_1])$,
II block in $w^{\max} = ([y, a_3]_-, [x', n_2]_-, [1, a_1]_-)$, II block in $\theta^{\max} = ([y + 1, a_3]_-, [x' - 1, n_2 + 1]_-, [1, a_1 - 1]_-)$,
III block in $w^{\max} = [a_1 + 1, x' - 1]_-$, III block in $\theta^{\max} = ([a_2 + 1], [a_1 + 1, x' - 2]_-)$.
In Case 2 of **??**, we have (with notations as in **??**),
I block in $w^{\max} = ([x, a_2]_-, [p, a_1]_-)$, I block in $\theta^{\max} = ([x + 1, a_2]_-, [p - 1, a_1]_-)$,
II block in $w^{\max} = ([y, a_3]_-, [1, p - 1]_-)$, II block in $\theta^{\max} = ([y + 1, a_3]_-, [x - 1, x]_-, [1, p - 2]_-)$,
III block in $w^{\max} = ([a_2 + 1, y - 1]_-, [a_1 + 1, x - 1]_-)$, III block in $\theta^{\max} = ([a_2 + 1, y]_-, [a_1 + 1, x - 2]_-)$.

For $\tau \in W$, we shall denote the associated Schubert variety by $X(\tau)$.

Theorem 5.8. *Sing* $X(w^{max}) = X(\theta^{max})$

Proof. The result follows from Theorems **??**, **??**. □

We now show that the Conjecture 6.2 holds for $X := X(w^{\max})$.

Proposition 5.9. *A bad pattern in* w^{max} *has to be of Type II.*

Proof. If possible, assume that there is a Type I bad pattern d, b, c, a in $w^{\max}$, where $a < b < c < d$. We have, d belongs to the I or II block in $w^{\max}$.

(1) Let d belong to the I block in $w^{\max}$. This implies that $b \in$ I or II block.

Let b belong to the I block in which case we have c does not belong to the I block. In Case 1 of **??**, we have, $[b, d] \subset$ I block in $w^{\max}$, and this contradicts the facts that $b < c < d$ and c does not belong to the I block. Hence our assumption is wrong and the result follows. In Case 2 of **??**, we have, $d \geq x,\ p \leq b \leq a_1$. Hence we obtain $a \leq p$, and hence a, c should belong to the II block in $w^{\max}$. This in turn implies that $c \in [y, a_3]$, which is not possible since $c < d$, and $d < y$. The result follows from this.

Let now b belong to the II block in $w^{\max}$ in which case we have c, a should belong to the III block. In Case 1 of **??**, we have, $[a, c]$ is contained in the III block which is not possible, since $a < b < c$ and b does not belong to the III block. In Case 2 of **??**, we have, $b \in [1, p-1]$, since $b < d < y$. But now, $a \geq a_1 + 1\ (> p-1)$ (since a belongs to the III block), which contradicts the fact that $b > a$.

(2) Let d belong to the II block in $w^{\max}$. This implies that b belongs to the II block, and c, a belong to the III block. In Case 1 of **??**, we have, $[a, c] \subset$ III block, which is not possible, since $a < b < c$ and b does not belong to the III block. In Case 2 of **??**, we have, $b \in [1, p-1]$ necessarily (since $b < c < y$) in which case, we obtain $a > b$ (since $a \geq a_1 + 1 > p - 1$), which is not possible. □

Proposition 5.10. *There do exist Type II bad pattern in* w^{max}. *Further for any Type II bad pattern* c, d, a, b *in* w^{max}, *where* $a < b < c < d$, *we have,* $c \in [x, a_2],\ d \in [y, a_3],\ a \in [1, u],\ b \in [a_1 + 1, v]$, *where* $x =$ *or* $> n_2 + 1$ *according as* $n_1 =$ *or* $> k_1$, $y =$ *or* $> a_2 + 1$ *according as* $n_3 =$ *or* $> k_2$, $u = a_1$ *(respectively* $p - 1$*) in Case 1 (respectively Case 2) of* **??**, $v = (x' - 1)$ *(respectively* $x - 1$*) in Case 1 (respectively Case 2) of* **??**.

Proof. We have a Type II pattern in $w^{\max}$, namely, $c = x,\ d = y,\ a = u,\ b = v$ in $w^{\max}$ (here, x, y, u, v are as in Proposition **??**).

Let now c, d, a, b be a Type II pattern in $w^{\max}$. We have c belongs to the I block, d, a belong to the II block, and b belongs to the III block (necessarily). We distinguish the following two cases:

Case 1. $n_2 \geq n_1 + n_3$.

In this case we have $n_1 = k_1$, and clearly $c \in [n_2 + 1, a_2]$. This implies $d \in [y, a_3],\ a \in [1, a_1],\ b \in [a_1 + 1, x' - 1]$.

Case 2. $n_2 < n_1 + n_3$.

Again clearly $c \in [x, a_2]$. This implies $d \in [y, a_3],\ a \in [1, p-1],\ b \in [a_1 + 1, x - 1]$. □

Corollary 5.11. *With notations as in Proposition* **??**, *we have,* w^{max} *is non-vexillary.*

Proof. This follows (in view of Proposition **??**) from the definition of a vexillary permutation; recall (cf. [?]) that a permutation $(a_1, \cdots, a_n)$ is vexillary if it avoids the pattern c, d, a, b, where $a < b < c < d$. □

Lemma 5.12. (1) $Q_{w^{max}} = P_{w^{max}} = Q$.

(2) $\overline{B\theta^{max}B}$ *is* Q-Q *is stable , i.e., stable for multiplication on the left and right by* Q.

Proof. The assertions follow from the definition of $w^{\max}$ and $\theta^{\max}$ □

Theorem 5.13. *The Conjecture 6.2* **??** *holds for* w^{max}.

Proof. We first observe that $\theta^{\max} \in F_{w^{\max}}$; for, corresponding to the Type II pattern $c = x,\ d = y,\ a = u,\ b = v$ in $w^{\max}$ (here, x, y, u, v are as in Proposition **??**), we have the bad pattern a, c, b, d in $\theta^{\max}$. Let us denote $\theta^{\max}$ by ϕ. Let w_1 (resp. τ) be the element of S_n obtained from ϕ (resp. $w^{\max}$) by replacing a, c, b, d (resp. c, d, a, b) respectively by c, d, a, b (resp. a, c, b, d). Then clearly $\tau \le \phi$, and $w_1 \le w$. Further, $\overline{B\theta^{\max}B}$ is Q-Q stable (cf. Lemma **??**). Thus $\theta^{\max} \in F_{w^{\max}}$.

Let now $\tau_1 \in F_{w^{\max}}$.

We have an occurrance in τ_1 of the form (a, c, b, d), $a < b < c < d$, corresponding to the occurrance (c, d, a, b) in $w^{\max}$ (cf. Proposition **??**). The fact that $\tau_1 \in F_{w^{\max}}$ in particular implies that $\tau_1 \in W_Q^{\max}$, and this in turn implies that a belongs to the I block, c, b belong to the II block, and d belongs to the III block in τ_1, i.e., $a \in \tau_1^{(a_1)},\ b, c \in \tau_1^{(a_2)}$. Let w_1 be the element of S_n obtained from τ_1 by replacing a, c, b, d respectively by c, d, a, b. We have $w_1 \le w^{\max}$ (by definition of $F_{w^{\max}}$).

Let i (respectively j) be such that $w_1^{(a_1)}(i) \le a_1,\ w_1^{(a_1)}(i+1) > a_1,\ w_1^{(a_1)}(j) = c$. Note that $i \ge a_1 - k_1$ (since $w_1 \le w^{\max}$ and $w^{(a_1)}(a_1 - k_1) \le a_1$), $j \ge i + 1$ (since $c \ge x\,(= a_2 + 1 - k_1) > n_1\,(= a_1)$). Now, $\tau_1^{(a_1)}$ is obtained from $w_1^{(a_1)}$ by replacing c by a, where note that $a \le a_1$ (cf. Proposition **??**). From this we obtain $\tau_1^{(a_1)}(i+1) = w_1^{(a_1)}(i) \le a_1$ and hence

$$\tau_1^{(a_1)}(a_1 - k_1 + 1) \le a_1 \tag{1}$$

(note that $a_1 - k_1 + 1 \le i + 1$). Let k (respectively l) be such that $w_1^{(a_2)}(k) \le a_2,\ w_1^{(a_2)}(k+1) > a_2,\ w_1^{(a_2)}(l) = d$. Note that $k \ge a_2 - k_2$ (since $w_1 \le w^{\max}$ and $w^{(a_2)}(a_2 - k_2) \le a_2$), $l \ge k + 1$ (since $d \ge y > a_2$ (cf. Proposition **??**)). Now, $\tau_1^{(a_2)}$ is obtained from $w_1^{(a_2)}$ by replacing d by b, where note that $b \le a_2$. From this we obtain $\tau_1^{(a_2)}(k+1) = w_1^{(a_2)}(k) \le a_2$ and hence

$$\tau_1^{(a_2)}(a_2 - k_2 + 1) \le a_2 \tag{2}$$

(note that $a_2 - k_2 + 1 \le k + 1$). Now (1) and (2) together with Remark **??** imply that $\tau_1 \le \theta^{\max}$. The required result follows from this. $\square$

Example: Let $\mathbf{n} = (\mathbf{2}, \mathbf{3}, \mathbf{2})$. We have $n = 7,\ Q = P_2 \cap P_5$. Further, $V(1,2),\ V(2,1)$ are the two irreducible components of V. Let w_1, w_2 be the elements of W^Q such that $V(1,2) \cong Y_Q(w_1),\ V(2,1) \cong Y_Q(w_2)$. We have,

$$w_1^{(2)} = (2,5),\ w_1^{(5)} = (1, 2\,5\,6\,7),$$

$$w_2^{(2)} = (4,5),\ w_2^{(5)} = (1, 2\,4, 5\,7).$$

Let θ_1, θ_2 be the elements of W^Q such that $V(0,1) \cong Y_Q(\theta_1),\ V(1,0) \cong Y_Q(\theta_2)$. We have,

$$\theta_1^{(2)} = (1,2),\ \theta_1^{(5)} = (1, 2\,4, 5\,7),$$

$$\theta_2^{(2)} = (2,5),\ \theta_2^{(5)} = (1, 2\,3, 4, 5).$$

Further, we have,

$$w_1^{\max} = (5276143),\ \theta_1^{\max} = (2175463),$$

$$w_2^{\max} = (5472163),\ \theta_2^{\max} = (5243176).$$

Note that w_1 (resp. w_2) is non-vexillary with $5, 6, 1, 4$ (resp. $4, 7, 2, 3$) as a Type II occurance. Note also that the occurance $1, 5, 4, 6$ in θ_1 (resp. $2, 4, 3, 7$ in θ_2) corresponds to the occurance $5, 6, 1, 4$ in w_1 (resp. $4, 7, 2, 3$ in w_1).

References

[1] N. Bourbaki, *Groups et Algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968.
[2] V. Lakshmibai and P. Magyar, *Degeneracy schemes, Schubert varieties and Quiver varieties*, to appear in International Math. Research Notices.
[3] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in $SL(n)/B$*, Proc. Indian Acad. Sci. (Math. Sci.), **100**, (1990), 45-52.
[4] V. Lakshmibai and C. S. Seshadri, *Geometry of G/P-II*, Proc. Ind. Acad. Sci, 87(1978), 1-54.
[5] A. Lascoux and M.P. Schutzenberger, *Schubert polynomials and Littlewood-Richardson rule*, Letters in Math. Phys., vol 10 (1985), 111-124.
[6] C. Musili and C.S. Seshadri, *Schubert varieties and the variety of complexes*, Arithmetic and Geometry, vol II, Progress in Math. 36,(1983), 329-359.