MTH U341 Conservative vector fields homework
Due Monday, Nov. 17

Let $\vec{F}$ be the vector field in 3-space

$$\vec{F}(x,y,z) = <y^2+z^2, 2xy+z^2+3y^2, 2yz-2z+2xz>.$$

1. Find a potential function for $\vec{F}$.

$\vec{F} = \nabla f$ means $f_x = y^2+z^2$, $f_y = 2xy+z^2+3y^2$, $f_z = 2yz-2z+2xz$. Thus

$$f(x,y,z) = \int y^2+z^2\,dx = xy^2+xz^2+g(y,z).$$

Then $f_y = 2xy+z^2+3y^2$ yields

$$2xy + g_y(y,z) = 2xy+z^2+3y^2$$

or $g_y(y,z) = z^2+3y^2$, so

$$g(y,z) = \int z^2+3y^2\,dy = yz^2+y^3+h(z)$$

and $f = xy^2+xz^2+yz^2+y^3+h(z)$. Then $f_z = 2yz-2z+2xz$ yields

$$2xz+2yz+h'(z) = 2yz-2z+2xz$$

or $h'(z) = -2z$, so $h(z) = -z^2$ is a solution. This gives

$$f(x,y,z) = xy^2+xz^2+yz^2+y^3-z^2.$$

2. Let C be the curve parametrized by

$$\vec{r}(t) = <t^5-\sqrt{t(1-t^2)}, \sin(\pi t)+2\cos(\pi t), e^{t^3}\sin(\pi t)+2t>; \quad 0 \le t \le 1.$$

Find $\int_C \vec{F}\cdot d\vec{r}$.

Instead of parametrizing the integral, use the fundamental theorem

$$\int_{{}_A C_B} \nabla f \cdot d\vec{r} = f(B) - f(A).$$

$A = \vec{r}(0) = (0,2,0)$ and $B = \vec{r}(1) = (1,-2,2)$. Thus

$$\int_C \vec{F}\cdot d\vec{r} = xy^2+xz^2+yz^2+y^3-z^2\Big|_{(0,2,0)}^{(1,-2,2)} = (4+4-8-8-4)-8 = -20.$$

3. Is $\vec{F}$ path-independent? In other words, if A and B are points in 3-space and C_1 and C_2 are two paths from A to B, is $\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_2}\vec{F}\cdot d\vec{r}$? Explain your answer.

By part (1), $F = \nabla f$, where $f = xy^2+xz^2+yz^2+y^3+z^2$. The fundamental theorem for line integrals of vector fields tells us that for C any path starting at A and ending at B, we have

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(B) - f(A).$$

As $f(B)-f(A)$ does not depend on the choice of path C, only the starting point A and ending point B, $\vec{F}$ is therefore path-independent.