Calculus 3-An overview

This sheet gives an overview of the main themes we have covered during the semester, organized by topic rather than chronologically. For a more detailed review of the course material, you should look over the six review sheets put out during the semester.

The basic object of study in a first calculus course is a real-valued function of one real variable:

$$f : D \to \mathbb{R}; \quad D \text{ a subset of } \mathbb{R},$$

studied through the operations of differentiation and integration. In this course, we build on what we know of functions of one variable to study real-valued and vector-valued functions of 1 or more variables:

$$F : D \to \mathbb{R}^n; \quad D \text{ a subset of } \mathbb{R}^m,$$

with m, n usually 1,2 or 3. For instance, a real-valued function of several variables

$$f : D \to \mathbb{R}; \quad D \text{ a subset of } \mathbb{R}^m,\ m = 2, 3, \ldots$$

a vector-valued function of 1 variable (that is, a parametrized curve in $\mathbb{R}^2$ or $\mathbb{R}^3$)

$$\vec{r} : D \to \mathbb{R}^n; \quad D \text{ a subset of } \mathbb{R},\ n = 2, 3$$

a vector-valued function of 2 variables (that is, a parametrized surface in in $\mathbb{R}^3$)

$$\vec{r} : D \to \mathbb{R}^n; \quad D \text{ a subset of } \mathbb{R}^2,\ n = 3$$

or vector fields on regions in $\mathbb{R}^2$ or $\mathbb{R}^3$:

$$\vec{F} : D \to \mathbb{R}^n; \quad D \text{ a subset of } \mathbb{R}^n, n = 2, 3.$$

Vectors and vector algebra: Vectors give a convenient way to work with $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$ for $n \geq 4$.

We talked about vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ as describing a length and a direction. For working with vectors, it is convenient to use coordinates: move the vector so that the starting point is the origin, and the coordinates of the end-point gives the coordinates of the vector. To distinguish a vector from a point, we write a vector in coordinates as

$$\vec{v} = <a, b> \text{ for } v \text{ a vector in } \mathbb{R}^2$$

or

$$\vec{v} = <a, b, c> \text{ for } v \text{ a vector in } \mathbb{R}^3.$$

The *length* of a vector $\vec{v} = <a, b, c>$ is

$$|\vec{v}| = \sqrt{a^2 + b^2 + c^2}.$$

We have some basic operations with vectors: vector addition, scalar multiplication and dot product:

$$<x, y, z> + <x', y', z'> = <x + x', y + y', z + z'>$$
$$k \cdot <x, y, z> = <kx, ky, kz>$$
$$<x, y, z> \cdot <x', y', z'> = xx' + yy' + zz'$$

with similar operations for vectors in $\mathbb{R}^2$.

Using these operations all vectors in $\mathbb{R}^3$ can be expressed in terms of the *standard basis vectors*

$$\vec{i} = <1,0,0>, \ \vec{j} = <0,1,0>, \ \vec{k} = <0,0,1>$$

via

$$<x,y,z> = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}.$$

We have the *cross product* of vectors in $\mathbb{R}^3$:

$$<a,b,c> \times <a',b',c'> = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ a' & b' & c' \end{vmatrix}.$$

Geometrically, the dot product is

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta \tag{1}$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. Two vectors $\vec{v}$, $\vec{w}$ are *orthogonal* if $\vec{v} \cdot \vec{w} = 0$.

The cross product $\vec{v} \times \vec{w}$ is determined geometrically by the conditions

(1) $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}||\sin\theta|$.
(2) $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and to $\vec{w}$.
(3) the direction of $\vec{v} \times \vec{w}$ is determined by the right-hand rule.

Applications. The vector operations were used to: give equations of lines and planes in 3-space, find normal vectors to a plane, parametrize a line.

We also used vector-valued functions to parametrize curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ and surfaces in $\mathbb{R}^3$, and to define vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (more about these later). If a curve in $\mathbb{R}^3$ parametrized by a vector-valued function

$$\vec{r} : [a,b] \to \mathbb{R}^3; \quad \vec{r}(t) = <x(t), y(t), z(t)>$$

we view $\vec{r}(t)$ as the *position vector* of a particle moving in space. The derivative $\vec{r}'(t) = <x'(t), y'(t), z'(t)>$ is the *velocity vector* and the second derivative $\vec{r}''(t) = <x''(t), y''(t), z''(t)>$ is the *acceleration vector* of the moving particle. The velocity vector will be tangent to the curve C parametrized by $\vec{r}$, so we have the *unit tangent vector*

$$\vec{T}(t) = \frac{1}{|\vec{r}'(t)|}\vec{r}'(t) \tag{2}$$

to C.

For a parametrized surface S, with parametrizing function

$$\vec{r}(u,v) = <x(u,v), y(u,v), z(u,v)>$$

the two partial derivatives $\vec{r}_u(u,v)$, $\vec{r}_v(u,v)$ will be tangent to S, so the cross product $\vec{r}_u \times \vec{r}_v$ will be a *normal vector* to S. This gives us the *unit normal vector*

$$\vec{n} = \frac{1}{|\vec{r}_u \times \vec{r}_v|}\vec{r}_u \times \vec{r}_v \tag{3}$$

to S. This will be useful in computing surface integrals.

Derivatives: Generalizing the idea of the derivative of a function of 1 variable:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

we have various types of derivatives of functions of several variables. These are used to study the original function, just as for functions of one variable.

• *Partial derivatives.* For a function $f(x,y)$, we can consider f as a function of x, with y a "constant", giving us the partial derivative with respect to x

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}. \tag{4}$$

Similarly, we have the partial derivative with respect to y

$$\frac{\partial f}{\partial y}(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}. \tag{5}$$

Similar formulas define the partial derivatives for functions of 3 or more variables.

Just as for functions of 1 variable, taking $\Delta x, \Delta y$ to be small rather than finding the limit allows one to approximate partial derivatives if, for instance, the function f is given numerically or graphically.

The usual rules for differentiation of functions of one variable allow us to compute partial derivatives, where we just treat the "other" variables as constants.

• *Tangent plane and linear approximations.* Taking $\Delta x, \Delta y$ to be small gives the linear approximation formula

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot \Delta x + \frac{\partial f}{\partial y}(a,b)\cdot \Delta y. \tag{6}$$

Writing $\Delta x = x - a$, $\Delta y = y - b$, this approximation formula gives us the equation of the tangent plane to the graph $z = f(x,y)$ at the point $(a, b, f(a,b))$:

$$z - f(a,b) = \frac{\partial f}{\partial x}(a,b)\cdot (x-a) + \frac{\partial f}{\partial y}(a,b)\cdot (y-b). \tag{7}$$

• *The chain rule.* The chain rule for functions of one variable extends to several variables: If $x = x(t)$, $y = y(t)$, $f = f(x,y)$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} \tag{8}$$

In this equation, you should substitute $(x,y) = (x(t), y(t))$ into the partial derivatives so that the right-hand side is a function of t. There is also a several variable version: if $x = x(u,v)$, $y = y(u,v)$, $f = f(x,y)$, then

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}, \tag{9}$$

with a similar formula replacing u with v; as above, you should substitute $(x,y) = (x(u,v), y(u,v))$ after taking the partial derivatives. There are also versions of the chain rule for functions f of 3 or more variables.

• *Directional derivatives and the gradient.* The partial derivatives extend to the operation of taking the derivative in any direction. For a unit vector $\vec{u}$, the *directional derivative* of a function f in the direction $\vec{u}$ at $\vec{x}$ is

$$D_{\vec{u}}(f)(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}.$$

$D_{\vec{u}}(f)(\vec{x})$ tells us the rate of change of f at $\vec{x}$, if one moves in the direction $\vec{u}$. Using the chain rule, we compute $D_{\vec{u}}(f)(\vec{x})$ in terms of the partial derivatives as

$$D_{\vec{u}}(f)(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} \tag{10}$$

where $\nabla f(\vec{x})$ is the *gradient vector*:

$$\nabla f(x,y) = <\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)>$$

for $f = f(x,y)$ and

$$\nabla f(x,y,z) = <\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)>$$

for $f = f(x,y,z)$. Similar formulas hold for functions of more than three variables.

• *Max/min.* Just as for functions of one variable, we use derivatives to find maxima and minima of functions of several variables.

A *critical point* of a function f is point p with

$$\nabla f(p) = 0$$

that is, all the partial derivatives of f vanish at p. Critical points are useful because

(11)
If a function f on a region R has a local maximum/local minimum at a point p, then p is a critical point of f.

Thus, to find the maximum/minimum values of a function f on a region R, we use the following procedure:

(1) Find the critical points of f that are in R.
(2) Look for maxima/minima of f on the boundary of R.

Step 2 is another max/min problem, but in one dimension lower, so eventually this process stops. One method to handle step 2 is to parametrize the boundary, giving one or more new max/min problems in fewer variables. Another method is to use Lagrange multipliers, discussed below.

Just as for functions of 1 variable, there is a *2nd derivative test* that we use to classify the behavior of f near a critical point. For a function $f(x,y)$, define

(12) $$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2.$$

The second derivative test is

(13)
If p is a critical point of $f(x,y)$, then

(1) *If $D(p) < 0$, then f has a saddle point at p*
(2) *If $D(p) > 0$ and $f_{xx}(p) > 0$ (or $f_{yy}(p) > 0$), then f has a local minimum at p*
(3) *If $D(p) > 0$ and $f_{xx}(p) < 0$ (or $f_{yy}(p) < 0$), then f has a local maximum at p*

• *Lagrange multipliers.* This is a method to find the max/min of a function f, subject to a *constraint* $g = k$. One solves the Lagrange multiplier equation

(14) $$\nabla f = \lambda \nabla g$$

together with the constraint equation $g = k$. This usually gives a finite collection of points, and one then evaluates f one these candidates to find the max/min.

Integrals: Integrals of a function of one variable are defined as a limit of Riemann sums. They are used to calculate things like the area under a graph, mass of a non-homgeneous straight wire, etc. We generalize this to functions of two and three variables, giving us double integrals over plane regions and triple integrals over solid regions.

• *Double and triple integrals as Riemann sums.* To define the double integral of a function $f(x,y)$ over a plane region R, divide up R into a grid of small rectangles Δx by Δy in size. In the ijth rectangle, choose a *sample point* (x_{ij}^*, y_{ij}^*) (if R is not itself a rectangle, just let the little rectangles extend outside R and define $f(x,y) = 0$ if (x,y) is not in R). Form the *Riemann sum* $\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$. The double integral is defined as the limit of Riemann sums

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y. \tag{15}$$

For reasonable regions and functions, this limit exists and does not depend on the choice of the smaple points. For triple integrals, one makes a similar construction, using little cubes instead of little rectangles.

From this definition, one can use the double integral or triple integral to compute lots of useful quantities. Basically anything that arises as a product of some quantity (a *density*) with area or volume, in case the quantity is constant, gets replaced with an integral if the density varies.

Example: density and mass. If a plane region R has constant density ρ grams/cm^2, and area A, then the mass is $\rho \cdot A$ grams. If R has varying density $\rho(x,y)$ grams/cm^2, then over a small rectangle of area dA at the point (x,y), the mass of the small rectangle, dm is given by

$$dm = \rho(x,y)\,dA$$

Adding up these bits of mass over R gives the mass of R, that is

$$m = \iint_R dm = \iint_R \rho(x,y)\,dA.$$

□

Similarly, the area of R is $\iint_R dA$, the mass of a solid D is $\iiint_S \rho\,dV$, the volume of the solid lying below a graph $z = f(x,y)$ and above a region R in the x-y plane is $\iint_R f(x,y)\,dA$, etc.

• *Iterated integrals.* You can compute double and triple integrals by doing a series of usual 1-variable integrals. For a double integral, you can choose the order of integration:

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{B(x)}^{T(x)} f(x,y)\,dy \right] dx = \int_c^d \left[\int_{L(y)}^{R(y)} f(x,y)\,dx \right] dy. \tag{16}$$

Here, $T(x)$ and $B(x)$ are the y-coordinates of the "top" and "bottom" of the the region R over x, and $L(y)$, $R(y)$ are the x-coordinates of the "left" and "right" edges of R, along fixed y. Similar formulas give triple iterated integrals for triple integrals.

• *Other coordinates.* You can sometimes calculate a double or triple integral more easily in an other coordinate system, such as polar, cylindrical or spherical coordinates, especially if the region of integration has a circular, cylindrical or spherical symmetry. You make the following substitutions:

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$.
Cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dV = r\,dr\,d\theta\,dz$.
Spherical coordinates: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$,
$z = \rho\cos\phi$, $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

evaluates f one these candidates to find the max/min.

Vector calculus: The fundamental theorem of calculus can be written as

$$\int_a^b f'(x)\,dx = f(b) - f(a),$$

expressing the relation between the two basic operations of differentiation and integration. The section on vector calculus extends this relation to functions of several variables, using double and triple integrals, as well as further generalizations of the integral: line integrals and surface integrals.

• *Vector fields.* This is the mathematical version of a force field (think of gravity) or velocity field (think of wind velocity). Mathematically speaking a *vector field on a domain D in* $\mathbb{R}^2$ is a vector-valued function

$$\vec{F} : D \to \mathbb{R}^2$$

Explicitly, $\vec{F}(x,y) = <P(x,y), Q(x,y)>$, with P and Q two usual functions on D. A vector field on a domain D in $\mathbb{R}^3$ is similarly a vector-valued function

$$\vec{F}(x,y,z) : D \to \mathbb{R}^3,$$

explicitly, $\vec{F}(x,y,z) = <P(x,y,z), Q(x,y,z), R(x,y,z)>$, with P, Q, R functions on D.

• *Conservative vector fields.* A very important way of constructing vector fields is by taking a function f on a domain D (in $\mathbb{R}^2$ or $\mathbb{R}^3$) and taking the gradient of f:

$$\vec{F}(\vec{x}) = \nabla f(\vec{x})$$

where $\vec{x} = (x,y)$ if D is a domain in $\mathbb{R}^2$ and $\vec{x} = (x,y,z)$ if D is a domain in $\mathbb{R}^3$. A vector field of this form is called *conservative*, and the function f is called a *potential function* for $\vec{F}$.

• *Line integrals.* If C is a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, and f is a function on C, define the *line integral with respect to arc length* $\int_C f\,ds$ as a limit of Riemann sums

$$\int_C f\,ds = \lim_{\substack{n\to\infty \\ \Delta s_i \to 0}} \sum_{i=1}^n f(p_i^*)\Delta s_i$$

where we divide up C into n sub-curves, the ith one having length Δs_i, and choose a sample point on each little piece, with p_i^* the chosen point on the ith piece. This definition shows that we can compute the length of C, and, given a density function

ρ on C, things like the mass or center of mass of C, as line integrals with respect to arc length. Explicitly

$$\text{arc length of } C = \int_C ds$$
$$\text{mass of } C = \int_C \rho\, ds$$

etc. (for more examples, see §13.2).

We can also integrate a vector field $\vec{F}$ over C, using the line integral with respect to $d\vec{r}$. For C a plane curve, $d\vec{r} = <dx, dy>$, $\vec{F} = <P, Q>$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_C <P, Q> \cdot <dx, dy> = \int_C P\, dx + Q\, dy$$

we have a similar expression of the line integral of a vector field for a curve in $\mathbb{R}^3$. In order to make sense out of a line integral of a vector field, C must be an *oriented* curve, that is, a curve with a direction. The line integral $\int_C \vec{F} \cdot d\vec{r}$ is defined as a Riemann sum, similar to the line integral $\int_C f\, ds$, except that we replace the length Δs_i with the change in (x, y), $<\Delta x_i, \Delta y_i>$:

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\substack{n\to\infty \\ \Delta x_i, \Delta y_i \to 0}} \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*) \cdot <\Delta x_i, \Delta y_i>.$$

The orientation is necessary, since moving along the curve in the opposite direction would change $<\Delta x_i, \Delta y_i>$ to $<-\Delta x_i, -\Delta y_i>$.

Interpretation. In case $\vec{F}$ is a force field, the line integral $\int_C \vec{F} \cdot d\vec{r}$ computes the *work* done by the force field on a particle moving along the path C.

• *Computing line integrals by parametrization.* To compute the line integrals, you need to parametrize the path of integration C by a vector valued function $\vec{r}(t)$, as t goes from a starting point a to an ending point b. We discuss the case of line integrals on plane curves; the case of curves in $\mathbb{R}^3$ is similar.

For line integrals with respect to arc length, you substitute $\vec{r}(t) = <x(t), y(t)>$ for the variable $\vec{x} = <x, y>$ (for curves in $\mathbb{R}^3$, $\vec{r}(t) = <x(t), y(t), z(t) = <x, y, z>$) and substitute

$$ds = |d\vec{r}| = |\vec{r}\,'(t)|\, dt = \sqrt{x'(t)^2 + y'(t)^2}\, dt \tag{17}$$

so

$$\int_C f\, ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\, dt.$$

For line integrals of vector fields, you also substitute:

$$d\vec{r} = \vec{r}\,'(t)\, dt = <x'(t), y'(t)>\, dt.$$

giving

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

Equivalently, substitute $dx = x'(t)\, dt$, $dy = y'(t)$ in the integral $\int_C P\, dx + Q\, dy$.

• *The fundamental theorem for vector fields.* The fundamental theorem for integrals in one variable generalizes for line integrals of vector fields as follows:

Let C be a path from A to B in the domain of a function f. Then

$$\int_{{}_AC_B} \nabla f \cdot d\vec{r} = f(B) - f(A). \tag{18}$$

Thus, for the conservative vector field $\vec{F} = \nabla f$, the potential function f acts like an anti-derivative.

• *Path independence and conservative vector fields.* A fundamental concept for vector fields is the notion of path independence of line integrals. For a vector field $\vec{F}$, we take two points A and B in the domain of $\vec{F}$ and consider the line integrals

$$\int_{{}_AC_B} \vec{F} \cdot d\vec{r}$$

for C a path from A to B in the domain of $\vec{F}$. The line integrals of $\vec{F}$ are called *path independent* if the value $\int_{{}_AC_B} \vec{F} \cdot d\vec{r}$ only depends on A and B, not the choice of path C from A to B.

It follows from the fundamental theorem for vector fields that a conservative vector field has this path independence property. In fact, the two properties are equivalent:

Let $\vec{F}$ be a vector field on a domain D. Then $\vec{F}$ is conservative on D if and only if all the line integrals of $\vec{F}$ are path independent.

• *Conservative vector fields on plane domains.* We consider a vector field $\vec{F}$ on an open plane domain D and ask the question: How do you tell if $\vec{F}$ is conservative? Well, if you can find a potential function for $\vec{F}$, then you are done, but this may be difficult. Notice that if $\vec{F} = <P, Q>$ is conservative with potential function f, then $<P, Q> = <f_x, f_y>$, so

$$\frac{\partial P}{\partial y} = f_{xy} = f_{yx} = \frac{\partial Q}{\partial x}$$

thus

If $\vec{F} = <P, Q>$ is conservative, then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

One can show by examples that there are vector fields satisfying this condition, but which are not conservative. The simplest test for whether a vector field is conservative is:

Let $\vec{F} = <P, Q>$ be a vector field on a simply connected open domain D in the plane. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

then $\vec{F}$ is conservative.

Warning: If the domain D is $\vec{F}$ is not simply connected, it does not mean that

$\vec{F}$ is not conservative, only that the above test for $\vec{F}$ to be conservative fails.

• *Finding a potential function for a conservative vector field.* If $F = <P, Q>$ is conservative, you may be able to find a potential function by successive "partial integration". For example, suppose P and Q are defined on all of $\mathbb{R}^2$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Since $\mathbb{R}^2$ is simply connected, $\vec{F}$ is conservative. If f is a potential function, then

$$\nabla f = <f_x, f_y> = <P, Q>$$

so

$$P = f_x, \; Q = f_y$$

See the homework handouts and quiz 5 for examples of how to use repeated integration to solve these equations for f.

• *Green's theorem.* Our next generalization of the fundamental theorem of calculus is:

Let R be a plane region. $\vec{F} = <P, Q>$ a vector field on R. Let C be the boundary of R, where we orient C by going in the counter-clockwise direction is C is an exterior boundary component of R, and going in the clockwise direction if C is an interior boundary component of R. Then

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \int_C \vec{F} \cdot d\vec{r}. \tag{19}$$

Green's theorem is useful for changing some "difficult" line integrals into "easy" double integrals and vice versa. Especially important is the case $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$; this makes the double integral zero, and thus the line integral over the boundary of R is zero as well. If the boundary has two components C_1, C_2, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \pm \int_{C_2} \vec{F} \cdot d\vec{r},$$

with the sign depending on the orientations for C_1 and C_2.

• *Surface integrals.* Just as line integrals generalize integrals in one variable, surface integrals generalize double integrals. For a surface S in $\mathbb{R}^3$ and a function f on S, we have the integral

$$\iint_S f \, dS$$

defined by dividing S up into small pieces, choosing sample points, forming a Riemann sum and taking a limit, just as for double integrals. This type of surface integral computes surface area:

$$\text{area } S = \iint_S dS$$

or mass, given a density function ρ on S

$$\text{mass } S = \iint_S \rho \, dS.$$

If we wish to integrate a vector field on S, we need an *orientation* on S, this being a continuously varying unit normal vector $\vec{n}$ on S. Define the vector area element $d\vec{S}$ as

$$d\vec{S} = \vec{n} \, dS$$

If $\vec{F}$ is a vector field on S, then $\vec{F} \cdot \vec{n}$ is a function on S and we can integrate

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS.$$

An integral of this type is call a *flux integral*: If $\vec{F}$ is the velocity vector field of a flowing gas or liquid, then $\iint_S \vec{F} \cdot d\vec{S}$ computes the total rate of flow (volume of gas per unit time) through S.

• *Computing surface integrals by parametrization.* Let

$$<x(u,v), y(u,v), z(u,v)> = \vec{r}(u,v)$$

be a parametrization of a surface S by a plane region R. Using the parametrization $\vec{r}$ we transform a surface integral over S into a double integral over R. You just substitute: $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$, S gets replaced with R and dS, $d\vec{S}$ are given by

$$dS = |r_u(u,v) \times r_v(u,v)|\, du\, dv \tag{20}$$

$$d\vec{S} = r_u(u,v) \times r_v(u,v)\, du\, dv \tag{21}$$

As it is sometimes useful, we note that

$$\vec{n} = \frac{1}{|r_u(u,v) \times r_v(u,v)|} r_u(u,v) \times r_v(u,v). \tag{22}$$

Warning: The formula for $d\vec{S}$ and $\vec{n}$ use the orientation of S that comes from the parametrization you are using. If you are already given an orientation of S, this one may agree with the one coming from the parametrization, or it may be its negative. To check you just need to check at one point. If they agree, fine, if not, just compute the surface integral by the substitutions given above and put in a minus sign in front of the integral to account for the difference in orientation.

• *curl and div.* To express further generalizations of the fundamental theorem, we need two operations on vector fields. For this, we introduce the vector operator

$$\nabla = <\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}>.$$

Let $\vec{F}$ be a vector field. The *curl* and *divergence* of $\vec{F}$ are

$$\text{curl } \vec{F} = \nabla \times \vec{F} \tag{23}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} \tag{24}$$

Explicitly, fo r $\vec{F} = <P, Q, R>$ the curl is

$$\begin{aligned} \text{curl } <P,Q,R> &= \nabla \times <P,Q,R> \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= <\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}>. \end{aligned}$$

You take the determinant by letting the partial derivatives operate on the functions when they are "multiplied" together in the determinant. The divergence of $<P, Q, R>$ is

$$\begin{aligned} \text{div } <P, Q, R> &= \nabla \cdot <P, Q, R> \\ &= <\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}> \cdot <P, Q, R> \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \end{aligned}$$

There are relations among div, curl and grad:

$$\text{curl grad } f = 0 \tag{25}$$

$$\text{div curl } \vec{F} = \vec{0} \tag{26}$$

The first of these tells us that if $\vec{F}$ is a conservative vector field on an open domain D in $\mathbb{R}^3$, then curl $\vec{F} = 0$. Just as in the plane, there are vector fields on some domains D with zero curl, but which are not conservative. However, for vector fields on $\mathbb{R}^3$, the situation is easier to control:

If $\vec{F}$ is a vector field on $\mathbb{R}^3$ and if curl $\vec{F} = 0$, then $\vec{F}$ is conservative.

Just as for vector fields in the plane, you can find a potential function for a conservative vector field on $\mathbb{R}^3$, solving the equations

$$f_x = P, \ f_y = Q, \ f_z = R$$

by successive partial integration, if you can do the integrals.

• *Stokes' theorem.* This generalizes Green's theorem from plane regions to surfaces.

If S is an oriented surface, the orientation vector $\vec{n}$ puts an orientation on the boundary curve C as follows: View $\vec{n}$ as telling you which direction is pointing "up" from S. Looking at S from "above", you orient the boundary curve C just as for plane regions: external components of C go counterclockwise and internal components go clockwise.

Having oriented the boundary of S, Stokes' theorem is:
Let S be an oriented surface with oriented boundary curve C, $\vec{F}$ a vector field on S. Then

$$\int\int_S \textit{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}. \tag{27}$$

You get Green's theorem back as a special case: orient a plane region R using the vector $\vec{k}$. If $\vec{F} = <P(x, y), Q(x, y), 0>$, then

$$\begin{aligned} \text{curl } \vec{F} \cdot d\vec{S} &= <0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}> \cdot \vec{k} \, dA \\ &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \end{aligned}$$

You apply Stokes' theorem essentially as you do Green's theorem. For examples, see the text and the handout on surface integrals

• *The divergence theorem.* Just as Green's theorem and Stokes' theorem relate integrals over surfaces with integrals over curves, the divergences theorem relates integrals over solids with integrals over surfaces. If S is the boundary surface of a solid region D, you orient S by taking the normal vector pointing *away* from D. The divergence theorem is

Let D be a solid region in $\mathbb{R}^3$, $\vec{F}$ a vector field on D, S the oriented boundary surface of D. Then

$$\iiint_D \mathit{div}\, \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}. \tag{28}$$

Just as for Green's theorem, the divergence theorem is most powerful if div $\vec{F} = 0$.

For applications, see the text and the handout on surface integrals.