MTH U341 Triple integrals homework

Due Monday, Nov. 10

1. Let D be the half-washer $1 \leq x^2 + y^2 \leq 9$, $y \geq 0$, and let E be the solid region above D and below the graph $z = 10 - x^2 - y^2$. Suppose E is made of a material with varying density y gms/cm^3. Compute the mass of E; use cylindrical coordinates to compute the integral.

mass=$\iiint_E y\, dV$. In cylindrical coordinates E is $0 \leq z \leq 10 - r^2$, $1 \leq r \leq 3$, $0 \leq \theta \leq \pi$, $dV = dz r dr d\theta$, $y = r\sin\theta$. The integral is

$$\begin{aligned}
\iiint_E y\, dV &= \int_0^\pi \int_1^3 \int_0^{10-r^2} r^2 \sin\theta dz\, dr\, d\theta \\
&= \int_0^\pi \int_1^3 (10 - r^2) r^2 \sin\theta\, dr\, d\theta \\
&= \int_0^\pi [(10/3)r^3 - (1/5)r^5] \sin\theta|_{r=1}^3\, d\theta \\
&= \int_0^\pi (574/15)\sin\theta\, d\theta \\
&= -(574/15)\cos\theta|_0^\pi \\
&= -(574/15)(-1-1) = 1148/15 \text{ gms.}
\end{aligned}$$

2. Let E the the hollow hemisphere $1 \leq x^2 + y^2 + z^2 \leq 4$, $z \geq 0$. If the density is z gms/cm^3, compute the mass of E. Use spherical coordinates. *Hint:* $\int \sin t \cos t\, dt$ can be found by using the substitution $u = \sin t$.

mass=$\iiint_E z\, dV$. In spherical coordinates, E is $1 \leq \rho \leq 2$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$. $z = \rho\cos\phi$ and $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. The mass is

$$\begin{aligned}
\iiint_E z\, dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} (1/4)\rho^4 \sin\phi\cos\phi|_1^2\, d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} (15/4)\sin\phi\cos\phi|_1^2\, d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (15/4)u\, du\, d\theta \text{ with } u = \sin\phi, du = \cos\phi\, d\phi \\
&= \int_0^{2\pi} (15/8)u^2|_0^1\, d\theta \\
&= 2\pi(15/8) = 15\pi/4 \text{ gms.}
\end{aligned}$$

3. Let E be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$ with density x gms/cm^3. Find the mass of E.

The plane through the points $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$ is given by a linear equation $z = ax + by + c$. Putting in $(x,y) = (0,0)$, $(1,0)$ and $(0,2)$ gives $c = 3$, $a = -3$, $b = -3/2$, so the tetrahedron is the solid region under the graph $z = 3 - 3x - (3/2)y$ and above the triangle R in the x-y plane, with vertices $(0,0)$, $(1,0)$, $(0,2)$. The top of this triangle is given by $y = 2 - 2x$. The mass is thus

$$\begin{aligned}
\iiint_E x\,dV &= \int_0^1 \int_0^{2-2x} \int_0^{3-3x-(3/2)y} x\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{2-2x} x(3-3x-(3/2)y)\,dy\,dx \\
&= \int_0^1 3x(1-x)(2-2x) - (3/4)x(2-2x)^2\,dx \\
&= \int_0^1 3x - 6x^2 + 3x^3\,dx \\
&= (3/2)x^2 - 2x^3 + (3/4)x^4|_0^1 = 1/4 \text{ gms.}
\end{aligned}$$