1. Let C be the curve $x^2+y^2=4$, $x \geq 0$, $y \geq 0$. Suppose C represents the path followed by a particle P as P moves from the point $(2,0)$ to the point $(0,2)$ in the plane. The particle moves through a force field $\vec{F}$ given by $\vec{F}(x,y) = <x-y, x>$ kg. m/sec^2, and x and y are in meters.

a. Write down a line integral over C that computes the work W done by $\vec{F}$ on P.

Work $= \int_C \vec{F} \cdot d\vec{r} = \int_C <x-y, x> \cdot <dx, dy> = \int_C (x-y)\,dx + x\,dy$.

b. Compute the integral by parametrizing C.

Parametrize C by $x = 2\cos t$, $y = 2\sin t$, $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$, $0 \leq t \leq \pi/2$. This gives

$$\begin{aligned}\int_C (x-y)\,dx + x\,dy &= \int_0^{\pi/2} (2\cos t - 2\sin t)(-2\sin t) + 2\cos t(2\sin t)\,dt \\ &= 4\int_0^{\pi/2} -\cos t \sin t + \sin t^2 + \cos^2 t\,dt \\ &= 4\int_0^{\pi/2} -\cos t\sin t + 1\,dt = -4\int_0^{\pi/2} \cos t \sin t\,dt + 4\int_0^{\pi/2} dt.\end{aligned}$$

The second integral is $4(\pi/2) = 2\pi$. Do the first integral by substituting $u = \sin t$, $du = \cos t\,dt$, so

$$\begin{aligned}-4\int_0^{\pi/2} \cos t \sin t\,dt &= -4\int_0^1 u\,du \\ &= -2u^2|_0^1 = -2.\end{aligned}$$

Thus

$$\int_C (x-y)\,dx + x\,dy = 2\pi - 2 \text{ Joules.}$$

2. You are given the following data: $\vec{F}(x,y) = <P(x,y), Q(x,y)>$ is a vector field defined for $(x,y) \neq (0,0)$ and the functions P and Q satisfy the equation

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

In addition, you are given that

$$\int_{C'} P\,dx + Q\,dy = 5,$$

where C' is the unit circle $x^2+y^2=1$, oriented in the counter-clockwise direction.

a. Let $D = \{(x,y) \text{ in } \mathbb{R}^2, (x,y) \neq (0,0)\}$, that is, D is the plane with the origin $(0,0)$ removed. Let A and B be points in D. Are line integrals $\int_{{}_A C_B} \vec{F} \cdot d\vec{r}$ independent of the choice of path C from A to B? If yes, explain why, if no, given an example of points A, B and two paths C_1, C_2 from A to B with $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$

The line integrals $\int_{{}_A C_B} \vec{F} \cdot d\vec{r}$ are not independent of the choice of path C from A to B. For an example, let C_1 be the path from $(1,0)$ to $(-1,0)$ going along the top of the circle C', and let C_2 be the path from $(1,0)$ to $(-1,0)$ going along the bottom of the circle C'. Then $C' = C_1 - C_2$, so

$$\begin{aligned} 5 &= \int_{C'} P\,dx + Q\,dy \\ &= \int_{C_1 - C_2} P\,dx + Q\,dy \\ &= \int_{C_1} P\,dx + Q\,dy - \int_{C_2} P\,dx + Q\,dy \end{aligned}$$

so $\int_{C_1} P\,dx + Q\,dy \neq \int_{C_2} P\,dx + Q\,dy$. As another example, let $C_1 = C'$, considered as a path from $(1,0)$ to $(1,0)$, and let C_2 be the path that just stays at the point $(1,0)$. Then $\int_{C_2} P\,dx + Q\,dy = 0$, but $\int_{C_1} P\,dx + Q\,dy = 5$, so the two integrals are not the same, and $\vec{F}$ is not path-independent.

b. Let $D = \{(x,y) \text{ in } \mathbb{R}^2, (x,y) \neq (0,0)\}$, that is, D is the plane with the origin $(0,0)$ removed. Is $\vec{F}$ conservative on D? Explain.

Since the line integrals $\int_{{}_A C_B} \vec{F} \cdot d\vec{r}$ are not independent of the choice of path C from A to B, $\vec{F}$ is not conservative on D.

3. Now let $\vec{G}(x,y) = <P(x,y), Q(x,y)>$ be another vector field defined for all (x,y) in $\mathbb{R}^2$ and with

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

(P and Q are different from what they were in #2).

a. Is $\vec{G}$ conservative on $\mathbb{R}^2$? Is there a potential function $g(x,y)$ for $\vec{G}$ on all of $\mathbb{R}^2$. Explain.

Yes, $\vec{G}$ is conservative on $\mathbb{R}^2$: $\mathbb{R}^2$ is an open simply connected domain, and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

so $\vec{G}$ is conservative on $\mathbb{R}^2$.

By definition of the term "conservative", this means that there is a function $g(x,y)$ on $\mathbb{R}^2$ with $\nabla g(x,y) = \vec{G}(x,y)$, making g a potential function for $\vec{G}$ on all of $\mathbb{R}^2$.

b. Let $D = \{(x,y) \text{ in } \mathbb{R}^2, (x,y) \neq (0,0)\}$. Is $\vec{G}$ conservative on D? Explain.

Yes, $\vec{G}$ is conservative on D. Since $\vec{G}$ is conservative on $\mathbb{R}^2$ by part (a), there is a potential function g for $\vec{G}$, that is, g is a function on $\mathbb{R}^2$ with $\nabla g(x,y) = \vec{G}(x,y)$ for all (x,y) in $\mathbb{R}^2$. But then $\nabla g(x,y) = \vec{G}(x,y)$ for all (x,y) in D, so the restriction of g to D is a potential functor for $\vec{G}$ on D, and thus $\vec{G}$ is conservative on D.